Published by Institute of Physics Publishing for SISSA/ISAS
Received: *January 24, 2005*
Accepted: *January 30, 2005*

New instanton effects in supersymmetric QCD

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Abstract: In supersymmetric QCD with $\mathrm{SU}(N_c)$ gauge group and N_f flavors, it is known that instantons generate a superpotential if $N_f = N_c - 1$ and deform the moduli space of vacua if $N_f = N_c$. But the role of instantons has been unclear for $N_f > N_c$. In this paper, we demonstrate that for $N_f > N_c$, on the moduli space of vacua, instantons generate a more subtle chiral operator containing (for example) non-derivative interactions of $2(N_f - N_c) + 4$ fermions. Upon giving masses to some flavors, one can integrate out some fermions and recover the standard results for $N_f = N_c$ and $N_f = N_c - 1$. For $N_f = N_c$, our analysis gives, in a sense, a more systematic way to demonstrate that instantons deform the complex structure of the moduli space of vacua.

Keywords: Solitons Monopoles and Instantons, Supersymmetry and Duality, Supersymmetric Effective Theories, QCD.

JHEP01(2005)056

Contents

1. Introduction

Supersymmetric QCD (or SQCD) with gauge group SU(N_c) and N_f massless flavors has been extensively studied. (By a flavor, we mean a massless chiral multiplet transforming in the fundamental plus antifundamental representation.) In particular, partly because of holomorphy and the extensive symmetry group this theory possesses, many properties of its low energy vacuum structure are amenable to exact analysis. Yet the theory still displays a wealth of interesting non-perturbative phenomena, including generation of a superpotential [1]–[5], deformation of the complex structure of the moduli space and appearance of composite massless particles [6] (see also [7] for a related analysis), and electric-magnetic duality [8]. For some short reviews of some aspects of this story, see [9, 10].

The properties of the theory depend very much on N_f. For $N_f = 0$, it is believed that a discrete chiral symmetry is dynamically broken in this theory. The behavior for $N_f < N_c - 1$ can largely be deduced from this statement [1]. For $N_f = N_c - 1$, instantons generate a superpotential which lifts all flat directions on the moduli space $\mathcal{M}$ of supersymmetric vacua. For $N_f = N_c$, instantons do not generate a superpotential, but rather [6] deform the complex structure of the moduli space $\mathcal{M}$. For $N_f > N_c$, no superpotential is generated and $\mathcal{M}$ is undeformed; the important dynamical statements for $N_f > N_c$ concern the behavior at the origin of $\mathcal{M}$, where composite massless particles appear [6], including [8] gauge fields related to electric-magnetic duality if $N_f \geq N_c + 2$.

Though the important statements for $N_f > N_c$ involve the behavior at the origin in $\mathcal{M}$, it is also possible to explore the behavior far from the origin for this range of N_f. In that region, the theory has gauge instantons whose effects are computable, just as they are for $N_f = N_c - 1$ and $N_f = N_c$. One might well ask, "What are these instantons good for?" The purpose of our paper is to answer this question. To keep things simple, we concentrate on the case $N_c = 2$, but we believe that the story is similar for all N_c.

Our answer is that instantons for all $N_f \geq N_c - 1$ generate a chiral interaction, or F-term, on $\mathcal{M}$. For $N_f = N_c - 1$, this is the familiar instanton-induced superpotential. For $N_f = N_c$, the F-term that is generated is a four-fermion (or two-derivative) interaction on $\mathcal{M}$ which describes the familiar deformation of the complex structure of $\mathcal{M}$. In fact, understanding the complex structure deformation in this slightly novel way gives, possibly, a more systematic way to understand how it comes about, and which $\mathcal{N} = 1$ theories have such a deformation. For $N_f > N_c$, we get more exotic F-terms on $\mathcal{M}$ which generate, for example, interactions with $2(N_f - N_c) + 4$ fermions. These interactions produce no obvious qualitative effect in the physics as long as the N_f flavors are massless, but if one adds bare masses for some flavors, they induce the usual instanton effects for $N_f = N_c$ and $N_f = N_c - 1$.

F-terms of the type that appear in our analysis — we will call them multi-fermion F-terms — have been little discussed in $\mathcal{N} = 1$ supersymmetric theories in four dimensions. This partly accounts for the fact that their appearance in SQCD has not been noted until now. Interactions of this type are far more familiar in the closely related context of two-dimensional sigma models with $\mathcal{N} = (2, 2)$ supersymmetry (which can arise by compactification or dimensional reduction from four dimensions); in that context, the multi-fermion F-terms are associated with the generators of the (c, c) chiral ring, or equivalently the ring of local observables of the topological B-model.

We begin our analysis in section 2 with a general description of multi-fermion F-terms in $\mathcal{N} = 1$ supersymmetric effective actions. In section 3, we analyze, for $N_c = 2$, the relevant terms that are possible in the specific case of supersymmetric QCD. We show that, for each value of N_f, their structure is uniquely determined by the flavor symmmetries. Here we exploit the fact that for $N_c = 2$, the flavor symmetry is enhanced because both the quarks and the anti-quarks transform in the fundamental representation of the gauge group.

Finally, we show in section 4 that these multi-fermion F-terms are indeed generated in the low energy effective action of SQCD. We do this in three ways, each of which casts a different light on the origin of these unusual corrections. First, we perform a direct instanton computation as in [1] to show that the multi-fermion F-terms are generated. Second, in the special case that $N_c = 2$, $N_f = N_c + 1 = 3$, we show that they arise from a tree-level Feynman diagram computation in the Seiberg dual description of the theory. Third, we show that upon perturbing the theory by supersymmetric bare masses for some flavors, these interactions give rise by renormalization group flow to the standard results for $N_f = N_c$ and $N_f = N_c - 1$. This result generalizes to $N_f > N_c$ the standard renormalization group flow [6] from the instanton-induced deformation of moduli space for $N_f = N_c$ to the instanton-induced superpotential for $N_f = N_c - 1$. We expect that these analyses will have analogs for all N_c and N_f.

Although our focus in this paper is on four-dimensional gauge theory, multi-fermion F-terms also appear naturally in supersymmetric effective actions that describe four-dimensional compactifications of string theory. In fact, the present work began with our asking whether worldsheet instantons of the heterotic string can deform the complex structure of the moduli space of supersymmetric vacua, like gauge instantons in SQCD for $N_f = N_c$. The question is motivated by the fact that [11] such worldsheet instantons can sometimes generate a spacetime superpotential, like gauge instantons for $N_f = N_c - 1$. The answer to the question is positive, as we will explain, along with other results, in a separate paper. A closely related matter is the analysis [12, 13] of holomorphic anomalies in the low energy effective action of the heterotic string.

2. General remarks on multi-fermion F-terms

In this section, we describe the general structure of F-terms containing many fermions in $\mathcal{N} = 1$ supersymmetric effective actions. However, before discussing generalities, we will motivate our study of these interactions by considering a very specific and well-known example: the complex structure deformation of the moduli space $\mathcal{M}$ of vacua that occurs in SU(2) SQCD with two flavors.

When the distinction is important, we write $\mathcal{M}_{cl}$ for the classical approximation to the moduli space of supersymmetric vacua, and $\mathcal{M}$ for the exact quantum moduli space.

2.1 Example: SU(2) SQCD with four doublets

For $N_c = 2$, that is for gauge group SU(2), the fundamental and antifundamental representations coincide, so the SU(2) gauge theory with N_f flavors is more naturally described as a theory with $2N_f$ doublets, that is, with chiral multiplets transforming as $2N_f$ copies of the two-dimensional representation **2**. Since the term "flavors" is a little misleading for SU(2), we will write the number of doublets as $2n$. (In SU(2) gauge theory with chiral multiplets transforming as doublets, the number of such doublets must be even because of a global anomaly [14].)

To establish notation for the rest of the paper, we combine the matter fields into one chiral multiplet,

$$Q^i_a = q^i_a + \theta\psi^i_a + \cdots \,, \tag{2.1}$$

with $a = 1, 2$ being the color index for the **2** of the SU(2) gauge symmetry, and $i = 1, \ldots, 2n$ being the flavor index for the **2n** of the global SU($2n$) flavor symmetry. Of course, we have indicated in (2.1) the component expansion of Q^i_a, including a scalar field q^i_a and a Weyl fermion ψ^i_a.

We also introduce the gauge invariant, composite meson chiral superfield M^{ij}, given by

$$M^{ij} = \epsilon^{ab}\, Q^i_a Q^j_b \,. \tag{2.2}$$

The meson M^{ij} is clearly anti-symmetric in the flavor indices i and j and so transforms in the skew representation $\wedge^2(\mathbf{2n})$ of SU($2n$).

Using the mesons M^{ij}, we can succinctly describe the classical moduli space $\mathcal{M}_{cl}$ of supersymmetric vacua as being parametrized by arbitrary expectation values of M^{ij} subject to the constraint

$$M \wedge M = 0\,, \tag{2.3}$$

or more explicitly,

$$\epsilon_{i_1 j_1 i_2 j_2 \cdots i_n j_n}\, M^{i_1 j_1} M^{i_2 j_2} = 0\,. \tag{2.4}$$

This system of quadratic equations (2.3) simply enforces the condition that

$$\mathrm{rank}(M) \leq 2\,, \tag{2.5}$$

as follows from the definition (2.2) of M^{ij} as the skew product of two quark superfields.

Now, if the number of doublets is $2n = 4$, the classical constraint (2.3) reduces to a single quadratic equation

$$\epsilon_{i_1 j_1 i_2 j_2} M^{i_1 j_1} M^{i_2 j_2} = 0 \tag{2.6}$$

which must be satisfied by M^{ij}. Upon introducing suitable complex linear combinations m^I, $I = 1, \ldots, 6$, of the six independent components M^{ij}, $i, j = 1, \ldots, 4$, so as to diagonalize the nondegenerate quadratic form that appears on the left hand side of (2.6), the classical equation (2.6) becomes

$$\sum_{I=1}^{6} (m^I)^2 = 0\,. \tag{2.7}$$

The classical moduli space $\mathcal{M}_{cl}$ is thus smooth away from the origin. Its singularity at the origin is a signal of the unbroken gauge symmetry. The m^I transform in the vector representation of the SU(4) or SO(6) flavor symmetry of the SU(2) gauge theory with four doublets.

The classical moduli space $\mathcal{M}_{cl}$ whose structure we have just reviewed is deformed in the quantum theory [6] and does not coincide with the quantum moduli space of vacua $\mathcal{M}$. To describe this deformation, we introduce the usual holomorphic coupling scale Λ. Then, in the quantum theory, the moduli space $\mathcal{M}$ is described by the modified constraint

$$M \wedge M = \Lambda^4\,, \tag{2.8}$$

or equivalently, with $\epsilon \sim \Lambda^4$,

$$\sum_{I=1}^{6} (m^I)^2 = \epsilon\,. \tag{2.9}$$

Up to a multiplicative constant, the form of the deformation (2.8) is determined completely by the SU(4) flavor symmetry and dimensional analysis. Of course, as a result of the deformation, the singularity of $\mathcal{M}_{cl}$ at the origin is removed and $\mathcal{M}$ is a smooth complex manifold.

JHEP01(2005)056

Representing the deformation in the effective action. But precisely how does the deformation (2.8) appear in the effective action of SQCD? Is there a term of order ϵ in the low energy effective action whose presence signals this deformation?

In this very simple example, one way to implement the quantum deformation in the low energy effective theory is to introduce a massive field Σ and a superpotential $W = \Sigma(M{\wedge}M - \Lambda^4)$ into the effective action, which thus takes the form

$$S = \int d^4x\, d^4\theta\, K\left(M, \overline{M}; \Sigma, \overline{\Sigma}\right) + \int d^4x\, d^2\theta\, W + \text{c.c.}, \tag{2.10}$$

where K is the Kähler potential. Solving the equations for a critical point of W, we find that $\Sigma = 0$ and $M{\wedge}M = \Lambda^4$. In this description, the term in the effective action that signals the deformation is clearly an F-term, the term $\Delta W = -\Lambda^4\Sigma$ in the effective superpotential. If this term is dropped, the constraint reduces to the classical one $M \wedge M = 0$.

The description we have just given is useful for this particular example, but relies on being able to describe the moduli space $\mathcal{M}_{cl}$ and its deformation $\mathcal{M}$ in terms of unconstrained linear fields Σ and m^I, $I = 1, \ldots, 6$, together with a superpotential. There are seven of these fields in all, of which in any given vacuum (away from the origin) two, namely Σ and a linear combination of the m^I, are massive, while five components of m^I are massless and parametrize the moduli space. Obviously, our deformation $\Delta W = -\Lambda^4\Sigma$ depends on the massive fields. In an analogous but different example, we might be unable to usefully describe the moduli space in terms of a linear sigma model with a superpotential. How can we describe the deformation in a low energy effective action constructed only from the massless fields?

We can find the answer by integrating out the massive fields to convert ΔW into an effective interaction for massless fields only. In doing so, we work modulo D-terms and attempt to determine what F-terms are generated. This computation is both simple and instructive and we will perform it, along with an analogous computation in the theory with six doublets, in section 4.2.

However, for now it is useful to simply use supersymmetry to determine what F-terms are possible on $\mathcal{M}_{cl}$. At least away from the origin of $\mathcal{M}_{cl}$, this theory is intrinsically described as an $\mathcal{N} = 1$ supersymmetric, nonlinear sigma model governing maps $\phi : M^4 \longrightarrow \mathcal{M}_{cl}$ from Minkowski space M^4 to $\mathcal{M}_{cl}$.

From this perspective, the perturbative effective action is the usual sigma model action,

$$S = \int d^4x\, d^4\theta\, K\left(\Phi^i, \overline{\Phi}^{\bar{i}}\right). \tag{2.11}$$

Here Φ^i and $\overline{\Phi}^{\bar{i}}$ are chiral and anti-chiral superfields whose lowest components ϕ^i and $\overline{\phi}^{\bar{i}}$ are local holomorphic and anti-holomorphic coordinates on $\mathcal{M}_{cl}$,[1] and K is again the Kähler potential associated to some Kähler metric $ds^2 = g_{i\bar{i}}\, d\phi^i d\overline{\phi}^{\bar{i}}$ on $\mathcal{M}_{cl}$. The reason that we consider a sigma model whose target is $\mathcal{M}_{cl}$ is that this the low energy structure in perturbation theory. We want to know how this description may be modified by instantons, in other words, what F-term on $\mathcal{M}_{cl}$ may be induced by instantons.

[1] In this discussion, 'i' is not a flavor index but an index parametrizing local coordinates on $\mathcal{M}_{cl}$.

JHEP01(2005)056

Of course, we know something about the answer: this F-term must describe the deformation of $\mathcal{M}_{cl}$ into $\mathcal{M}$. So let us discuss what terms in the effective action of an $\mathcal{N}=1$ sigma model with a given target (in our case, $\mathcal{M}_{cl}$) describe a deformation of the complex structure of the target. We want to consider the deformation not extrinsically, as a modification of some algebraic equations describing the target, but intrinsically, as a modification of the $\overline{\partial}$ operator of the target.

In general, a deformation of the complex structure on $\mathcal{M}_{cl}$ is described as a change in the $\overline{\partial}$ operator on $\mathcal{M}_{cl}$ of the form

$$\overline{\partial}_{\bar{j}} \longmapsto \overline{\partial}_{\bar{j}} + \omega_{\bar{j}}{}^{i}\, \partial_i \,. \tag{2.12}$$

Here $\omega_{\bar{j}}{}^{i}$ is a representative of a Dolbeault cohomology class in $H^1(\mathcal{M}_{cl}, T\mathcal{M}_{cl})$, whose elements parametrize infinitesimal deformations of $\mathcal{M}_{cl}$. We use standard notation, with $T\mathcal{M}_{cl}$ and $\Omega^1_{\mathcal{M}_{cl}}$ denoting the holomorphic tangent and cotangent bundles of $\mathcal{M}_{cl}$.

We can equally well represent the change (2.12) in the $\overline{\partial}$ operator on $\mathcal{M}_{cl}$ as a change in the dual basis of holomorphic one-forms $d\phi^i$,

$$d\phi^i \longmapsto d\phi^i - \omega_{\bar{j}}{}^{i}\, d\overline{\phi}^{\bar{j}} \,. \tag{2.13}$$

As a result, under the deformation, the metric on $\mathcal{M}_{cl}$ changes as

$$g_{i\bar{i}}\, d\phi^i d\overline{\phi}^{\bar{i}} \longmapsto g_{i\bar{i}} \left(d\phi^i - \omega_{\bar{j}}{}^{i}\, d\overline{\phi}^{\bar{j}} \right) d\overline{\phi}^{\bar{i}} \,, \tag{2.14}$$

so that, upon deforming $\mathcal{M}_{cl}$, the metric picks up a component of type $(0,2)$ when written in the original holomorphic and anti-holomorphic coordinates. (Of course, there is also a complex conjugate term of type (2,0).)

Since we know how the metric on $\mathcal{M}_{cl}$ changes when $\mathcal{M}_{cl}$ is deformed, we can immediately deduce that the corresponding correction to the sigma model action is generally of the form

$$\delta S = \int d^4x\, d^2\theta\, \omega_{\bar{i}\bar{j}}\, \overline{D}\,\overline{\Phi}^{\bar{i}} \cdot \overline{D}\,\overline{\Phi}^{\bar{j}} = \int d^4x\, \omega_{\bar{i}\bar{j}}\, d\overline{\phi}^{\bar{i}}\, d\overline{\phi}^{\bar{j}} + \cdots \,, \tag{2.15}$$

with

$$\omega_{\bar{i}\bar{j}} = \frac{1}{2}\left(g_{i\bar{i}}\, \omega_{\bar{j}}{}^{i} + g_{i\bar{j}}\, \omega_{\bar{i}}{}^{i} \right) . \tag{2.16}$$

Here $\overline{D} \equiv \overline{D}_{\dot{\alpha}}$ is the usual spinor covariant derivative on superspace, and we have introduced the shorthand notation "$\cdot$" for the contraction of spinor indices (so for any two spinors η and ζ, $\eta \cdot \zeta$ is shorthand for $\eta_{\dot{\alpha}}\zeta^{\dot{\alpha}}$). We have also performed the fermionic integral over θ in (2.15), from which we see that the leading bosonic term reproduces the correction to the metric in (2.14).

Of course, the most important property of δS — and the primary motivation for this paper — is the fact that δS is an F-term. But δS is not a correction to the superpotential — it generates terms with two derivatives of bosons, or with four fermions. Because of the latter contribution, δS is a special case of what we call a multi-fermion F-term.

In contrast to a superpotential interaction, a deformation of the complex structure (of a smooth complex manifold, such as $\mathcal{M}_{cl}$ with the origin removed) is trivial locally. So locally on $\mathcal{M}_{cl}$, it must be possible to write δS in the form $\int d^4\theta(\ldots)$. As will become clear, this cannot be done globally on $\mathcal{M}_{cl}$, and it cannot be done even locally in a way that respects the SU(4) flavor symmetry. In that sense, δS is a non-trivial F-term.

We also note that this F-term is not manifestly supersymmetric, since the operator $\mathcal{O}_\omega = \omega_{\bar{i}\bar{j}}\,\overline{D}\overline{\Phi}^{\bar{i}}\cdot\overline{D}\overline{\Phi}^{\bar{j}}$ is not manifestly chiral. Rather, the chirality of $\mathcal{O}_\omega$ in the on-shell supersymmetry algebra determined by the unperturbed sigma model action S follows from the fact that $\omega_{\bar{j}}{}^{i}$ is annihilated by $\overline{\partial}$.

In section 2.2, we discuss more systematically the basic properties of multi-fermion F-terms such as δS.

Computing δS in SQCD. We have described in general what sort of term in the low energy effective action of an $\mathcal{N}=1$ sigma model describes the deformation of the complex structure of the target. We will now be more explicit for SU(2) gauge theory with four doublets.

For this purpose, we reconsider the extrinsic, algebraic description of the deformation of $\mathcal{M}_{cl}$, using the coordinates m^I. Instead of saying that the deformation changes the equation

$$\sum_I (m^I)^2 = 0 \tag{2.17}$$

to

$$\sum_I (m^I)^2 = \epsilon\,, \tag{2.18}$$

we want to work in a description in which the target space remains the same (away from the origin and perturbatively in ϵ) but a new interaction is generated.

It is possible to do this because away from the origin the quantum constraint (2.18) can be converted to the classical constraint (2.17) by a non-holomorphic change of variables. When the coordinates m^I satisfy the quantum constraint (2.18), the new coordinates

$$\widetilde{m}^I = m^I - \frac{\epsilon}{2}\,\frac{\delta^I_{\overline{J}}\,\overline{m}^{\overline{J}}}{\overline{m}m} \tag{2.19}$$

obey the classical constraint (2.17) to first order in ϵ. (We could work beyond first order, but this is not necessary.) Here $\overline{m}m = \sum_{I=1}^{6}|m^I|^2$, and in describing $\widetilde{m}^I$ we introduce the tensor $\delta^I_{\overline{J}}$ constructed from the SO(6) invariant tensors δ^{IJ} and $\delta_{I\overline{J}}$; in the language of SU(4), these tensors would be respectively $\epsilon^{ij}{}_{\bar{k}\bar{l}}$, ϵ^{ijkl}, and $(\delta_{i\bar{i}}\,\delta_{j\bar{j}} - \delta_{j\bar{i}}\,\delta_{i\bar{j}})$.

Thus, when the original coordinates m^I satisfy the quantum constraint, the new coordinates $\widetilde{m}^I$ satisfy the classical constraint, at least to leading order in ϵ,

$$\sum_{I=1}^{6}\left(\widetilde{m}^I\right)^2 = \mathcal{O}(\epsilon^2)\,. \tag{2.20}$$

The new coordinates $\widetilde{m}^I$ are obviously not holomorphic in the old complex structure on $\mathcal{M}_{cl}$, but we can find a new complex structure in which they are holomorphic. The deformation of the complex structure can be described as in (2.12) by correcting the $\overline{\partial}$ operator. From the requirement that the new coordinates $\widetilde{m}^I$ be holomorphic in the new complex structure, we have that

$$\left(\frac{\partial}{\partial \overline{m}^{\overline{J}}} + \omega_{\overline{J}}{}^{I}\,\frac{\partial}{\partial m^I}\right)\widetilde{m}^K = 0\,. \tag{2.21}$$

From this equation, we can directly solve for the tensor $\omega_{\overline{J}}{}^{I}$ in terms of the components m^I of M. We find, again to leading order in ϵ, that

$$\omega_{\overline{J}}{}^{I} = \frac{\epsilon}{2}\left(\frac{\delta^I_{\overline{J}}}{\overline{m}m} - \frac{\overline{m}^I m_{\overline{J}}}{(\overline{m}m)^2} - \frac{m^I \overline{m}_{\overline{J}}}{(\overline{m}m)^2}\right), \tag{2.22}$$

with indices raised and lowered with δ^{IJ} and $\delta_{I\overline{J}}$ as appropriate.

In this expression, only the first two terms in (2.22) arise directly from solving the equation (2.21). In fact, the last term in the expression for $\omega_{\overline{J}}{}^{I}\, d\overline{m}^{\overline{J}}\, \partial/\partial m^I$ vanishes identically when we restrict to $\mathcal{M}$, as on $\mathcal{M}$ we have the relation

$$0 = \sum_{I=1}^{6} \overline{m}^{\overline{I}} d\overline{m}^{\overline{I}} = \frac{1}{2} d\left(\sum_I (\overline{m}^{\overline{I}})^2\right). \tag{2.23}$$

We have included this trivial term in $\omega_{\overline{J}}{}^{I}$ just so that, upon lowering one index with the Kähler metric, the tensor $\omega_{\overline{I}\overline{J}}$ is manifestly symmetric.

Of course, we do not actually know the Kähler metric g on $\mathcal{M}_{cl}$, as appears implicitly in determining δS by converting the section ω of $\overline{\Omega}^1_{\mathcal{M}_{cl}} \otimes T\mathcal{M}_{cl}$ to a section of $\overline{\Omega}^1_{\mathcal{M}_{cl}} \otimes \overline{\Omega}^1_{\mathcal{M}_{cl}}$, as in (2.15) and (2.16). By symmetry, we do know that this metric must equal the metric on $\mathcal{M}_{cl}$ induced from the euclidean metric times a function of $\overline{m}m$, and asymptotically for large $\overline{m}m$ the metric must reduce to the classical metric describing canonical kinetic terms for underlying quarks in the ultraviolet regime of SQCD.

All of our expressions for the multi-fermion F-terms depend on the metric g. However, this dependence is irrelevant in the sense that the fundamental holomorphic object ω which represents a class in $H^1(\mathcal{M}, T\mathcal{M})$ and determines the existence of the multi-fermion F term does not depend on a choice of Kähler metric. Of course, the metric is known asymptotically, near infinity on $\mathcal{M}$, where it can be determined from the underlying classical field theory and asymptotic freedom.

We will now give a concrete formula for δS. Because of the dependence on g, we can present this formula in various ways. The most general approach, which also leads to the simplest expressions, is simply to leave g implicit, absorbing it into the index structure of $\omega_{\overline{I}\overline{J}}$ as we did in (2.16). This means that we simply use an unknown Kähler metric in raising and lowering indices. With this convention understood, from (2.15), (2.16), and (2.22), we see that δS takes the form

$$\delta S = \int d^4x\, d^2\theta\, \frac{\epsilon}{2}\left(\frac{\delta^{IJ}}{\overline{m}m} - \frac{\overline{m}^I\, m^J}{(\overline{m}m)^2} - \frac{m^I\, \overline{m}^J}{(\overline{m}m)^2}\right) \overline{D}\overline{m}_I \cdot \overline{D}\overline{m}_J\,. \tag{2.24}$$

Alternatively, this expression (2.24) is what results if we assume that g is the flat metric, so that we simply raise and lower indices with the Kronecker delta.

JHEP01(2005)056

On the other hand, because the mesons m^I and $\overline{m}^{\overline{I}}$ most naturally (that is in the classical theory) have dimension 2, the metric $g_{I\overline{I}}$ most naturally has dimension -2 (so that $ds^2 = g_{I\overline{I}} dm^I dm^{\overline{I}}$ has dimension two). As a result, the dimensional analysis of our expression in (2.24) is not transparent. Asymptotically on $\mathcal{M}$, the Kähler potential is known to be asymptotic to $K = \sqrt{\overline{m}m}$. With this knowledge, we can make the asymptotic form of the interaction more precise. In doing so, it is convenient to also make dimensional analysis manifest by simply using the Kronecker delta $\delta_{I\overline{I}}$ to raise and lower indices on m and $\overline{m}$, while writing factors of $\sqrt{\overline{m}m}$ explicitly. In this case, all components of m and $\overline{m}$ with indices up or down have dimension two. The asymptotic qform of the interaction δS then becomes

$$\delta S = \int d^4x\, d^2\theta\, \frac{\epsilon}{2\sqrt{\overline{m}m}} \left(\frac{\delta^{IJ}}{\overline{m}m} - \frac{\overline{m}^I\, m^J}{(\overline{m}m)^2} - \frac{m^I\, \overline{m}^J}{(\overline{m}m)^2} \right) \overline{D}\,\overline{m}_I \cdot \overline{D}\,\overline{m}_J\,. \tag{2.25}$$

Recalling that $\epsilon \sim \Lambda^4$, one can check directly that the naive dimensional analysis holds.

In the rest of the paper, we will mainly follow the first convention, as in (2.24), so that g appears only implicitly.

In terms of the components M^{ij} of M written using SU(4) flavor indices, as we will use in section 3, the expression (2.24) becomes

$$\delta S = \int d^4x\, d^2\theta\, \Lambda^4 \left(\frac{\epsilon^{i_1 j_1 i_2 j_2}}{\overline{M}M} - \frac{\epsilon^{i_1 j_1 kl}\, M^{i_2 j_2}\, \overline{M}_{kl}}{2\left(\overline{M}M\right)^2} - \frac{\epsilon^{i_2 j_2 kl}\, M^{i_1 j_1}\, \overline{M}_{kl}}{2\left(\overline{M}M\right)^2} \right) \overline{D}\,\overline{M}_{i_1 j_1} \cdot \overline{D}\,\overline{M}_{i_2 j_2}\,. \tag{2.26}$$

Here we take $\overline{M}M \equiv \frac{1}{2}\sum_{ij} \overline{M}_{ij} M^{ij}$. (The factor of 1/2 is included so that if the only nonzero components of M^{ij} are $M^{12} = -M^{21} = 1$, then $\overline{M}M = 1$. The factors of 1/2 in (2.26) relative to (2.24) arise from this convention and lead to the simple formula below.)

For future reference, we observe that up to a constant factor the expression in (2.26) can be written more compactly as

$$\delta S = \Lambda^4 \int d^4x\, d^2\theta\, \left(\overline{M}M\right)^{-2}\, \epsilon^{i_1 j_1 i_2 j_2}\, \overline{M}_{i_1 j_1} \left(M^{kl}\, \overline{D}\,\overline{M}_{i_2 k} \cdot \overline{D}\,\overline{M}_{l j_2} \right). \tag{2.27}$$

In section 3, we will show that this form (2.27) of the F-term is completely determined by symmetry and furthermore extends naturally to the case of SU(2) SQCD with $n > 2$ flavors.

2.2 Multi-fermion F-terms

Our description of the complex structure deformation in SQCD by means of a multi-fermion F-term may seem perverse, as the algebraic description of the deformation in (2.8) is so much simpler than (2.27). However, by phrasing this deformation as a multi-fermion F-term in an effective four-dimensional $\mathcal{N} = 1$ supersymmetric sigma model, we can see an immediate generalization to F-terms of even higher order.

To introduce this generalization, we begin by recalling that a four-dimensional sigma model with $\mathcal{N} = 1$ supersymmetry can be dimensionally reduced to a two-dimensional sigma model with $\mathcal{N} = (2,2)$ supersymmetry. Under this reduction, chiral operators in one sigma model map naturally to chiral operators in the other. The multi-fermion F-terms in four dimensions have better-known analogs in two dimensions.

In two dimensions, rings of chiral operators have been much studied [15]–[18] in the context of string theory and correspond to the rings of local observables in the topological A- and B-models. In fact — with the superpotential being a typical example — F-terms in four dimensions reduce to chiral observables of the B-model in two dimensions. The chiral operators of the B-model are in one-to-one correspondence with elements of $H^p(\mathcal{M}, \wedge^q T\mathcal{M})$.

To construct multi-fermion F-terms, we begin with a section ω of $\overline{\Omega}^p_{\mathcal{M}} \otimes \overline{\Omega}^p_{\mathcal{M}}$. (Were it not for the requirement of Lorentz-invariance, we could more generally start with a section of $\overline{\Omega}^p_{\mathcal{M}} \otimes \overline{\Omega}^q_{\mathcal{M}}$ for $p \neq q$.) Explicitly, ω is given by a tensor $\omega_{\bar{i}_1 \cdots \bar{i}_p \bar{j}_1 \cdots \bar{j}_p}$ that is antisymmetric in the $\bar{i}_k$ and also in the $\bar{j}_k$. Given such a tensor, we construct a possible term in the effective action that generalizes what we found in (2.15):

$$\begin{aligned} \delta S &= \int d^4x\, d^2\theta\, \omega_{\bar{i}_1 \cdots \bar{i}_p \bar{j}_1 \cdots \bar{j}_p} \left(\overline{D}\,\overline{\Phi}^{\bar{i}_1} \cdot \overline{D}\,\overline{\Phi}^{\bar{j}_1}\right) \cdots \left(\overline{D}\,\overline{\Phi}^{\bar{i}_p} \cdot \overline{D}\,\overline{\Phi}^{\bar{j}_p}\right), \\ &\equiv \int d^4x\, d^2\theta\, \mathcal{O}_\omega\,. \end{aligned} \tag{2.28}$$

To achieve Lorentz invariance, spinor indices are contracted here; for example, $(\overline{D}\,\overline{\Phi}^{\bar{i}_1} \cdot \overline{D}\,\overline{\Phi}^{\bar{j}_1})$ is an abbreviation for $(\overline{D}_{\dot\alpha}\overline{\Phi}^{\bar{i}_1}\, \overline{D}^{\dot\alpha}\overline{\Phi}^{\bar{j}_1})$. Furthermore, given the form of this operator, we can assume that ω is symmetric under the overall exchange of i's and j's.

Supersymmetry of $\mathcal{O}_\omega$. The interaction δS is not manifestly supersymmetric. For it to be supersymmetric, $\mathcal{O}_\omega$ must be chiral, that is, annihilated by the anti-chiral supersymmetries $\overline{Q}_{\dot\alpha}$. Even if δS is supersymmetric, it may represent a trivial F-term. Though written in (2.28) in the form $\int d^2\theta(\ldots)$, it may be that δS can be alternatively written $\int d^4\theta(\ldots)$, in other words as a D-term. This will be so if it is possible to write $\mathcal{O}_\omega = \{\overline{Q}_{\dot\alpha}, [\overline{Q}^{\dot\alpha}, V]\}$ for some V. If so, $\mathcal{O}_\omega$ is trivially chiral and $\delta S = \int d^4x\, d^4\theta\, V$.

To describe the chirality condition on $\mathcal{O}_\omega$, which will be no surprise from experience with the two-dimensional B-model, we first note that we can use the Kähler metric $g_{i\bar{i}}$ on $\mathcal{M}$ to raise either set of $\bar{i}$ or $\bar{j}$ indices on ω. The raised indices become holomorphic, so upon raising the indices, ω becomes interpreted as a section of $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$ in two distinct ways. By our assumption on the symmetry of ω, we find the same section of $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$ either way.

We now consider the action of the anti-chiral supercharges $\overline{Q}_{\dot\alpha}$ in the on-shell supersymmetry algebra of the unperturbed sigma model, so that we consider for simplicity only the linearized supersymmetry constraint on δS. Under the action of $\overline{Q}_{\dot\alpha}$, the component fields ϕ^i and ψ^i_β of Φ^i and the component fields $\overline{\phi}^{\bar{i}}$ and $\overline{\psi}^{\bar{i}}_{\dot\beta}$ of $\overline{\Phi}^{\bar{i}}$ transform as

$$\begin{aligned} \delta_{\dot\alpha}\phi^i &= 0\,, & \delta_{\dot\alpha}\overline{\phi}^{\bar{i}} &= \overline{\psi}^{\bar{i}}_{\dot\alpha}\,, \\ \delta_{\dot\alpha}\psi^i_\beta &= i\,\partial_{\dot\alpha\beta}\phi^i\,, & \delta_{\dot\alpha}\overline{\psi}^{\bar{i}}_{\dot\beta} &= -\Gamma^{\bar{i}}_{\bar{j}\,\bar{k}}\,\overline{\psi}^{\bar{j}}_{\dot\alpha}\,\overline{\psi}^{\bar{k}}_{\dot\beta}\,. \end{aligned} \tag{2.29}$$

Here Γ is the connection associated to the Kähler metric $g_{i\bar{i}}$ on $\mathcal{M}$. So long as we consider only the action of a single supercharge, we can without loss set Γ to zero by a suitable coordinate choice on $\mathcal{M}$.

By using the metric to interpret each set of anti-chiral fermions $\overline{\psi}^{\bar{i}}_{\dot{\beta}}$ for $\dot{\beta} = 1, 2$ as alternatively anti-holomorphic one-forms $d\overline{\phi}^{\bar{i}}$ or holomorphic tangent vectors $\partial/\partial\phi^i$, we see directly from (2.29) that the action of each of the two supercharges $\overline{Q}_{\dot{\alpha}}$ on $\mathcal{O}_\omega$ corresponds to the action of $\overline{\partial}$ on ω when ω is regarded as a section of $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$ in either of the two possible ways. Thus, the chirality constraint on $\mathcal{O}_\omega$ is simply the condition that ω be annihilated by $\overline{\partial}$. This result is familiar in the B-model.

Cohomology of $\mathcal{O}_\omega$. We must also impose an equivalence relation on the space of operators $\mathcal{O}_\omega$, such that $\mathcal{O}_\omega$ is considered trivial if δS is equivalent to a D-term. The condition we will get is closely related to the reduction to $\overline{\partial}$ cohomology in the B-model.

As a simple example, any perturbative correction δK to the Kähler form can be trivially rewritten as an F-term correction upon performing half the integral over superspace:

$$\begin{aligned}\int d^4x\, d^4\theta\, \delta K &= \int d^4x\, d^2\theta\, \overline{D}^2 \delta K\,,\\ &= \int d^4x\, d^2\theta\, \nabla_{\bar{i}}\nabla_{\bar{j}}\delta K \left(\overline{D}\,\overline{\Phi}^{\bar{i}} \cdot \overline{D}\,\overline{\Phi}^{\bar{j}}\right).\end{aligned} \tag{2.30}$$

In the second line, we have introduced the covariant derivative ∇ associated to the connection Γ in (2.29), and we have explicitly rewritten the chiral integrand in the form of an operator $\mathcal{O}_\omega$, with

$$\omega_{\bar{i}\bar{j}} = \nabla_{\bar{i}}\nabla_{\bar{j}}\delta K\,. \tag{2.31}$$

Even more generally, we must consider possible corrections to the effective action which involve integrals over three quarters of superspace[2] and are of the form

$$\begin{aligned}\delta S &= \int d^4x\, d^2\theta\, d\overline{\theta}_{\dot{\alpha}}\, \xi_{\bar{i}_2\cdots\bar{i}_p\,\bar{j}_1\cdots\bar{j}_p}\, \overline{D}^{\dot{\alpha}}\overline{\Phi}^{\bar{j}_1}\left(\overline{D}\,\overline{\Phi}^{\bar{i}_2}\cdot\overline{D}\,\overline{\Phi}^{\bar{j}_2}\right)\cdots\left(\overline{D}\,\overline{\Phi}^{\bar{i}_p}\cdot\overline{D}\,\overline{\Phi}^{\bar{j}_p}\right),\\ &\equiv \int d^4x\, d^2\theta\, d\overline{\theta}_{\dot{\alpha}}\, \mathcal{O}_\xi{}^{\dot{\alpha}}\,,\\ &= \int d^4x\, d^2\theta\, \nabla_{\bar{i}_1}\xi_{\bar{i}_2\cdots\bar{i}_p\,\bar{j}_1\cdots\bar{j}_p}\left(\overline{D}\,\overline{\Phi}^{\bar{i}_1}\cdot\overline{D}\,\overline{\Phi}^{\bar{j}_1}\right)\cdots\left(\overline{D}\,\overline{\Phi}^{\bar{i}_p}\cdot\overline{D}\,\overline{\Phi}^{\bar{j}_p}\right).\end{aligned} \tag{2.32}$$

Here ξ is a section of $\overline{\Omega}^{p-1}_{\mathcal{M}} \otimes \overline{\Omega}^p_{\mathcal{M}}$.

Because the correction in (2.32) has the same form as the F-term in (2.28), we must consider the chiral operators $\mathcal{O}_\omega$ as defined up to the equivalence

$$\mathcal{O}_\omega \sim \mathcal{O}_\omega + \left\{\overline{Q}_{\dot{\alpha}}, \mathcal{O}_\xi{}^{\dot{\alpha}}\right\}. \tag{2.33}$$

Mathematically, this equivalence becomes an equivalence relation on sections of $\overline{\Omega}^p_{\mathcal{M}} \otimes \overline{\Omega}^p_{\mathcal{M}}$,

$$\omega_{\bar{i}_1\cdots\bar{i}_p\,\bar{j}_1\cdots\bar{j}_p} \sim \omega_{\bar{i}_1\cdots\bar{i}_p\,\bar{j}_1\cdots\bar{j}_p} + \nabla_{[\bar{i}_1}\xi_{\bar{i}_2\cdots\bar{i}_p]\,\bar{j}_1\cdots\bar{j}_p} + \left(\bar{i}_k \leftrightarrow \bar{j}_k\right). \tag{2.34}$$

[2]We do not know of any actual examples of operators of this type that can be written as integrals over three quarters of superspace but not over all of superspace.

As we indicate, the term involving ξ is to be symmetrized like ω under the exchange of all pairs $\bar{i}_k \leftrightarrow \bar{j}_k$.

Because of this symmetrization, the equivalence relation implied by (2.34) on sections of $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$ is not the same as the usual equivalence relation in Dolbeault cohomology. Furthermore, since the corrections (2.32) arise from an integral only over three quarters of superspace, they are not supersymmetric unless we impose the (nontrivial) condition that $\overline{Q}^2$ annihilate the operator $\mathcal{O}_\xi{}^{\dot{\alpha}}$, which implies a corresponding constraint on the sections ξ which appear in (2.32) and (2.34).

We are unaware of a more standard mathematical description of this sort of cohomology, specific to the bundles $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$ on an arbitrary Kähler manifold, and we will not comment further on its general structure. Luckily, symmetries alone will suffice in section 3 to show that the operators $\mathcal{O}_\omega$ which we consider for SQCD cannot be written as integrals over three-fourths of superspace, much less all of it.

2.3 Adding a superpotential to the sigma model

Although we are most interested in SQCD with massless flavors, a useful technique to study this theory is to consider instead SQCD with massive flavors and to ask how various observables depend upon the mass parameters. Because these mass parameters appear in a superpotential, holomorphy serves as a powerful tool to constrain their appearance in the effective action. In section 4, we will apply exactly this technique as one way to compute the multi-fermion F-terms in SQCD.

More generally, we can consider adding any background superpotential W to the basic sigma model action,

$$S = \int d^4x\, d^4\theta\, K(\Phi^i, \overline{\Phi}^{\bar{i}}) + \int d^4x\, d^2\theta\, W(\Phi^i) + \text{c.c.}\,. \tag{2.35}$$

Because of the superpotential, the on-shell supersymmetry algebra of the sigma model is altered, and hence the chirality condition on $\mathcal{O}_\omega$ is also altered. This fact is fundamental to our study of the mass deformation of SQCD in section 4, so we pause to explain it here in the general setting.

In the new action (2.35), the on-shell variations under $\overline{Q}_{\dot{\alpha}}$ of the component fields ϕ^i, $\overline{\phi}^{\bar{i}}$, ψ^i_β, and $\overline{\psi}^{\bar{i}}_{\dot{\beta}}$ are now given by

$$\begin{aligned} \delta_{\dot{\alpha}}\phi^i &= 0\,, & \delta_{\dot{\alpha}}\overline{\phi}^{\bar{i}} &= \overline{\psi}^{\bar{i}}_{\dot{\alpha}}\,, \\ \delta_{\dot{\alpha}}\psi^i_\beta &= i\,\partial_{\dot{\alpha}\beta}\phi^i\,, & \delta_{\dot{\alpha}}\overline{\psi}^{\bar{i}}_{\dot{\beta}} &= -\Gamma^{\bar{i}}_{\bar{j}\,\bar{k}}\,\overline{\psi}^{\bar{j}}_{\dot{\alpha}}\,\overline{\psi}^{\bar{k}}_{\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}\,g^{\bar{i}i}\partial_i W\,. \end{aligned} \tag{2.36}$$

Because of the appearance of the one-form dW in the variation of $\overline{\psi}^{\bar{i}}_{\dot{\beta}}$ in (2.36), the action of the supercharges $\overline{Q}_{\dot{\alpha}}$ on $\mathcal{O}_\omega$ is no longer given geometrically by the action of $\bar{\partial}$ on ω. Instead, when $\overline{\psi}^{\bar{i}}_{\dot{\beta}}$ is interpreted as a holomorphic tangent vector $\partial/\partial\phi^i$, the term involving W corresponds geometrically to the interior product of $\partial/\partial\phi^i$ with the holomorphic one-form dW. So the $\bar{\partial}$ operator is now generalized to the operator

$$\delta = \bar{\partial} + \iota_{dW}\,, \tag{2.37}$$

JHEP01(2005)056

acting on sections of $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$. Here ι_{dW} denotes the operator on $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$ which acts by the interior product with the one-form dW. (In other words, ι_{dW} acts by removing $\overline{\psi}$ and replacing it with dW.) We note that because W is holomorphic, $\delta^2 = 0$. Thus, the first order chirality condition on the operator $\mathcal{O}_\omega$ becomes the requirement that δ annihilate ω.

A nice mathematical discussion of the cohomology theory associated to δ is given by Liu in [19], and applications to string theory are discussed in [20].

When ω is a section of $\overline{\Omega}^1_{\mathcal{M}} \otimes T\mathcal{M}$, then the modified chirality condition has a very direct geometric interpretation. In this case, the condition that $\delta\omega = 0$ implies that ω is annihilated separately by both the operators $\overline{\partial}$ and ι_{dW}. The latter condition implies that

$$\omega_{\overline{j}}^{\ i}\, \partial_i W = 0\,. \tag{2.38}$$

Since W is holomorphic, this condition is then equivalent to the condition that

$$\left(\overline{\partial}_{\overline{j}} + \omega_{\overline{j}}^{\ i}\, \partial_i\right) W = 0\,, \tag{2.39}$$

implying that the deformation of $\overline{\partial}$ represented by ω must preserve the holomorphy of W. More generally, if it is possible to modify W to a function $W + \Delta W$ that is holomorphic in the deformed complex structure, then $\omega + \Delta W$ is annihilated by δ.

3. Multi-fermion F-terms in SU(2) SQCD

Up to this point, we have discussed general properties of multi-fermion F-terms in an arbitrary $\mathcal{N} = 1$ sigma model. We now specialize our analysis to the particular case of SQCD. Our main goal in the rest of the paper, concentrating mainly on the example of gauge group SU(2), is to show that multi-fermion F-terms are generated in the effective action of SQCD.

To this end, we begin in this section by analyzing the constraints imposed by symmetries and holomorphy on the form of any multi-fermion F-term corrections in SU(2) SQCD. The case of SQCD with gauge group SU(2) is particularly simple due to the enhancement of the flavor symmetry. In this case, we fix the form of the operators $\mathcal{O}_\omega$ uniquely, and we demonstrate that they are nontrivial in the cohomology of $\overline{Q}_{\dot{\alpha}}$.

In the general case of SQCD with gauge group SU(N_c) and $N_f > N_c$ flavors, a similar analysis to determine the form of the operators $\mathcal{O}_\omega$ appears to be more complicated, since the geometry of the moduli space $\mathcal{M}$ itself is more complicated. However, the direct instanton computation of section 4.1 shows that such interactions arise for all N_c and $N_f \geq N_c - 1$. The other derivations in section 4 generalize in spirit.

In the case of SU(2) SQCD with $N_f = n$ flavors, we have already described algebraically the classical moduli space $\mathcal{M}$ as being parametrized by the mesons M^{ij}, subject to the system of quadratic equations $M \wedge M = 0$. This description of $\mathcal{M}$ has the virtue of being very succinct. However, we now give another description of $\mathcal{M}$ which makes its symmetry more apparent and consequently enables us to determine immediately the chiral operators $\mathcal{O}_\omega$ which arise from cohomology classes on $\mathcal{M}$.

JHEP01(2005)056

3.1 More about the geometry of $\mathcal{M}$

Since symmetries are of the utmost importance, we first review the symmetries of SU(2) SQCD with $N_f = n$ flavors. Besides the SU(2) color and SU($2n$) flavor symmetries, this gauge theory also possesses a non-anomalous U(1) $\mathcal{R}$-symmetry as well as an anomalous U(1) axial symmetry. Under these symmetries, the quark superfields Q^i_a, the mesons M^{ij}, and the holomorphic coupling scale Λ transform as follows:

$$\begin{array}{ccccc} & \mathrm{SU}(2)_c & \mathrm{SU}(2n) & \mathrm{U}(1)_A & \mathrm{U}(1)_{\mathcal{R}} \\ Q^i_a & \mathbf{2} & \mathbf{2n} & 1 & 1-\frac{2}{n} \\ M^{ij} & \mathbf{1} & \wedge^2(\mathbf{2n}) & 2 & 2\left(1-\frac{2}{n}\right) \\ \Lambda^{6-n} & \mathbf{1} & \mathbf{1} & 2n & 0 \end{array} \tag{3.1}$$

Here Λ^{6-n} is the standard instanton counting parameter. (In this one place, we denote the gauge group as SU(2)$_c$, to distinguish it from an unbroken SU(2) flavor group that will appear momentarily.)

We now describe $\mathcal{M}$ by considering the pattern of symmetry breaking around a fixed supersymmetric vacuum. Up to the action of the symmetries, any solution of the usual D-term equations takes the form

$$Q^i_a = \begin{pmatrix} v & 0 \\ 0 & v \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \equiv v\,\widehat{\delta}^i_a\,, \tag{3.2}$$

with v being an arbitrary complex number.

So long as v is non-zero, the expectation value of Q^i_a in (3.2) breaks the symmetry group in (3.1) down to a subgroup

$$\mathrm{SU}(2) \times \mathrm{SU}(2n-2) \times \mathrm{U}(1)'_A \times \mathrm{U}(1)'_{\mathcal{R}}\,. \tag{3.3}$$

The unbroken $\mathrm{SU}(2)\times\mathrm{SU}(2n-2)$ factor arises in the obvious way, and the unbroken U(1) axial and $\mathcal{R}$-symmetries arise from linear combinations of the corresponding generators in (3.1) with the diagonal flavor generator in the center of the subgroup

$$S(\,\mathrm{U}(2)\times\mathrm{U}(2n-2))\;\subset\;\mathrm{SU}(2n)\,. \tag{3.4}$$

Of course, the gauge group is completely Higgsed, and the massless fluctuations of the quarks Q^i_a about the point (3.2) decompose into two irreducible representations of the unbroken symmetry group (3.3), with

$$\begin{array}{ccccc} & \mathrm{SU}(2) & \mathrm{SU}(2n-2) & \mathrm{U}(1)'_A & \mathrm{U}(1)'_{\mathcal{R}} \\ \Phi^s_c & \mathbf{2} & \mathbf{2n-2} & \frac{n}{n-1} & \frac{n-2}{n-1} \\ \Phi & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \Lambda^{6-n} & \mathbf{1} & \mathbf{1} & 2n & 0 \end{array} \tag{3.5}$$

Here the singlet Φ describes a rescaling of M; the other fluctuations transform as an irreducible representation Φ_c^s of the unbroken symmetry, where $c = 1, 2$ is an index labelling the $\mathbf{2}$ of the unbroken SU(2) and $s = 3, \ldots, 2n$ is now an index labelling the $\mathbf{2n-2}$ of $\mathrm{SU}(2n-2)$. Throughout the paper, we will apply the convention that c, d, e, f refer to indices $1, 2$ of the unbroken SU(2), that s, t, u, v refer to indices $3, \ldots, 2n$ of the unbroken $\mathrm{SU}(2n-2)$, and that i, j, k, l run over all indices $1, \ldots, 2n$ of the full $\mathrm{SU}(2n)$ flavor symmetry. These massless fluctuations Φ and Φ_c^s represent local coordinates on $\mathcal{M}$, such as were used in section 2. Finally, for future reference in section 3.2 we have included in (3.5) the charges of Λ^{6-n}, which are identical to those in (3.1).

Because any solution of the D-term equations can be brought to the form (3.2) using the $\mathrm{SU}(2) \times \mathrm{SU}(2n)$ symmetry of SQCD, we see that the $\mathrm{SU}(2n)$ flavor symmetry acts transitively on the quotient of $\mathcal{M}$ minus the origin by the $\mathbb{C}^*$ action which scales v. We thus set $\widetilde{\mathcal{M}} = \mathcal{M} - \{0\}$, and we let B be this quotient of $\widetilde{\mathcal{M}}$ by $\mathbb{C}^*$.

Furthermore, our description of the symmetry breaking pattern in (3.3) is equivalent to the geometric observation that, at any non-zero v, the subgroup of $\mathrm{SU}(2n)$ which stabilizes the point corresponding to $Q_a^i = v\,\widehat{\delta}_a^i$ on $\mathcal{M}$ is $S(\,\mathrm{U}(2) \times \mathrm{U}(2n-2))$. Thus, we can describe B as a homogeneous (and in fact symmetric) space,

$$B = \mathrm{SU}(2n)/S(\,\mathrm{U}(2) \times \mathrm{U}(2n-2))\,. \tag{3.6}$$

To incorporate the value of v into our description of $\mathcal{M}$, we observe that the $\mathbb{C}^*$ action which scales v is the complexification of the $\mathrm{U}(1)_A$ symmetry in (3.1). This symmetry corresponds to the action of the central U(1) which lies in the stabilizer subgroup $S(\,\mathrm{U}(2) \times \mathrm{U}(2n-2))$ and whose generator mixes with the generator of $\mathrm{U}(1)_A$ under the symmetry breaking. Associated to this U(1) generator in $S(\,\mathrm{U}(2) \times \mathrm{U}(2n-2))$ is a corresponding homogeneous line bundle $\mathcal{L}$ — and hence a $\mathbb{C}^*$ bundle — over B. To specify $\mathcal{L}$, we simply note that the singlet field Φ transforms as a section of $\mathcal{L}$ and has charge $+2$ under the original $\mathrm{U}(1)_A$ symmetry, as Φ describes the rescaling of M^{ij}.

So, if we excise the singularity at the origin of $\mathcal{M}$, then $\widetilde{\mathcal{M}}$ can be globally described as this $\mathbb{C}^*$ bundle over the base B,

$$\mathbb{C}^* \longrightarrow \widetilde{\mathcal{M}} \xrightarrow{\pi} B\,. \tag{3.7}$$

A direct relationship now exists between the algebraic description of $\mathcal{M}$ in (2.3) and the intrinsic description of $\mathcal{M}$ in (3.7). To describe this relation, we consider the mesons M^{ij} modulo overall scaling, corresponding to the $\mathbb{C}^*$ action generated by $\mathrm{U}(1)_A$. Then the equations $M \wedge M = 0$ are the classical Plücker relations [21] which describe the Grassmannian $Gr(2, 2n)$ of complex two planes in $\mathbb{C}^{2n}$ as an algebraic subvariety of the projective space parametrized by M^{ij}.

On the other hand, this Grassmannian can also be described as a quotient,

$$Gr(2, 2n) = \mathrm{U}(2n)/(\,\mathrm{U}(2) \times \mathrm{U}(2n-2))\,, \tag{3.8}$$

which is equivalent to our description in (3.6) of the base B. Thus, the $\mathbb{C}^*$ bundle over B in (3.7) is simply the bundle associated to the affine cone over the Grassmannian $Gr(2, 2n)$

JHEP01(2005)056

with its Plücker embedding in projective space. Equivalently, the line bundle $\mathcal{L}$ arises as the pullback from the degree one bundle $\mathcal{O}(1)$ on projective space.

3.2 The new F-terms

With our thorough discussion of the symmetries of SQCD, we can immediately derive the form of any multi-fermion F-terms that might appear on $\mathcal{M}$. We perform our analysis in two steps: first locally, and then globally.

Local analysis. Locally, we construct the chiral operator $\mathcal{O}_\omega$ from the massless fluctuations described by Φ^s_c and Φ about the vacuum $Q^i_a = \widehat{\delta}^i_a$. Thus, in terms of the section ω of $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$, we only consider ω as restricted to the tangent space of $\mathcal{M}$ at this point.

Now, the operator $\mathcal{O}_\omega$ must be invariant under the symmetries $\mathrm{SU}(2) \times \mathrm{SU}(2n-2) \times \mathrm{U}(1)'_A$ and must have charge $+2$ under $\mathrm{U}(1)'_{\mathcal{R}}$ in (3.5). Furthermore, since we are only considering the corresponding section ω as restricted to the tangent space of a point in $\mathcal{M}$, we must construct $\mathcal{O}_\omega$ completely from the fermionic fields $\overline{D}_{\dot\alpha}\overline{\Phi}$ and $\overline{D}_{\dot\alpha}\overline{\Phi}^c_s$ which represent either one-forms or (by raising an index) tangent vectors to $\mathcal{M}$. From (3.5) we see that $\overline{D}_{\dot\alpha}\overline{\Phi}$ and $\overline{D}_{\dot\alpha}\overline{\Phi}^c_s$ have respective charges $+1$ and $+1/(n-1)$ under $\mathrm{U}(1)'_{\mathcal{R}}$. So just to make an operator of $\mathrm{U}(1)'_{\mathcal{R}}$ charge $+2$, we require that it contain either two copies of $\overline{D}_{\dot\alpha}\overline{\Phi}$, or one copy of $\overline{D}_{\dot\alpha}\overline{\Phi}$ and $n-1$ copies of $\overline{D}_{\dot\alpha}\overline{\Phi}^c_s$, or $2(n-1)$ copies of $\overline{D}_{\dot\alpha}\overline{\Phi}^c_s$.

We can immediately rule out the first possibility, necessarily of the form $\overline{D}\,\overline{\Phi}\cdot\overline{D}\,\overline{\Phi}$, since from (3.5) this operator is not charged under $\mathrm{U}(1)'_A$ and hence is not multiplied by any power of Λ, contradicting the fact that our operator must vanish in the appropriate weak coupling limit as well as the fact that we expect it to be generated by instantons. (A more detailed study shows that there are no non-trivial chiral operators of this type.) On the other hand, since the only tensors of $\mathrm{SU}(2) \times \mathrm{SU}(2n-2)$ which we can use to make invariants out of the fields $\overline{D}_{\dot\alpha}\overline{\Phi}^c_s$ are the anti-symmetric tensors ϵ_{cd} and $\epsilon^{s_1 t_1 \cdots s_p t_p}$ with $p = n-1$, we cannot make an invariant operator from one copy of $\overline{D}_{\dot\alpha}\overline{\Phi}$ and only $n-1$ copies of $\overline{D}_{\dot\alpha}\overline{\Phi}^c_s$.

We are left to consider the operator $\mathcal{O}_\omega$ which is made from $2(n-1)$ copies of $\overline{D}_{\dot\alpha}\,\overline{\Phi}^c_s$, of the form

$$\Lambda^{6-n}\,\epsilon^{s_1 t_1 \cdots s_p t_p}\,\epsilon_{c_1 d_1}\cdots\epsilon_{c_p d_p}\,\left(\overline{D}\,\overline{\Phi}^{c_1}_{s_1}\cdot\overline{D}\,\overline{\Phi}^{d_1}_{t_1}\right)\cdots\left(\overline{D}\,\overline{\Phi}^{c_p}_{s_p}\cdot\overline{D}\,\overline{\Phi}^{d_p}_{t_p}\right), \qquad p = n-1\,. \tag{3.9}$$

This operator is invariant under $\mathrm{SU}(2) \times \mathrm{SU}(2n-2)$ and carries charge $+2$ under $\mathrm{U}(1)'_{\mathcal{R}}$. The pattern of contractions of spinor indices is fixed by the fact that each expression in parentheses must be antisymmetric under exchanges of both the pairs (c,d) and (s,t) and must also obey Fermi statistics.

Also, we see from (3.5) that each fermion appearing in $\mathcal{O}_\omega$ carries charge $-n/(n-1)$ under $\mathrm{U}(1)'_A$, so the fermionic part of $\mathcal{O}_\omega$ carries axial charge $-2n$. The fact that $\mathcal{O}_\omega$ must be invariant under the axial symmetry then fixes the dependence on Λ. In particular, we see that the operator in (3.9) involves a single power of the instanton counting parameter Λ^{6-n} and so could arise as a one-instanton effect.

JHEP01(2005)056

So the local form of $\mathcal{O}_\omega$ is fixed completely by the symmetries, and moreover $\mathcal{O}_\omega$ has the correct dependence on Λ to be generated by instantons. Furthermore, in terms of the section ω of $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$, we see that the parameter p is related to the number of flavors n by $p = n - 1$. This fact is a special case of the relation $p = N_f - N_c + 1$ which must hold in $\mathrm{SU}(N_c)$ SQCD with N_f flavors. In the direct instanton computation in section 4, this relation follows most immediately by counting fermion zero modes in the instanton background.

A geometric remark on pullbacks from B. Because $\mathcal{O}_\omega$ only involves $\overline{D}\,\overline{\Phi}^c_s$ and not the singlet $\overline{D}\,\overline{\Phi}$, the section ω has only components along the base B, with no legs along the $\mathbb{C}^*$ fiber. Naively, one might have concluded that ω then arises as the pullback from a section of $\overline{\Omega}^{n-1}_B \otimes \wedge^{n-1} TB$ on B. Actually, the dependence of $\mathcal{O}_\omega$ on scaling of the quark superfields means that it is a pullback from a section of $\overline{\Omega}^{n-1}_B \otimes \wedge^{n-1} TB \otimes \mathcal{L}^k$ for some k.[3]

In fact, $k = -n$. Indeed, as we noted above, the fermionic part of $\mathcal{O}_\omega$ carries $\mathrm{U}(1)'_A$ charge $-2n$. As $\mathrm{U}(1)'_A$ differs from $\mathrm{U}(1)_A$ by a generator of $\mathrm{SU}(2n)$ under which $\mathcal{O}_\omega$ is invariant, this means that, if we omit the factor of Λ^{6-n} from (3.9), then $\mathcal{O}_\omega$ has $\mathrm{U}(1)_A$ charge $-2n$. Since the basic meson field M has $\mathrm{U}(1)_A$ charge 2, this means that $\mathcal{O}_\omega$ transforms as M^{-n} and ω can be regarded as a section of $\overline{\Omega}^{n-1}_B \otimes \wedge^{n-1} TB \otimes \mathcal{L}^{-n}$.

Consider a general scaling $M \to \lambda M$, $\overline{M} \to \overline{\lambda}\,\overline{M}$, for $\lambda \in \mathbb{C}^*$. Under this scaling, $\omega \to \lambda^{-n}\overline{\lambda}^0\,\omega = \lambda^{-n}\omega$. The fact that the exponent of $\overline{\lambda}$ is zero is implied by the fact that $\overline{\partial}\omega = 0$, and the fact that the exponent of λ is $-n$ is equivalent to the fact that ω is a section of $\overline{\Omega}^{n-1}_B \otimes \wedge^{n-1} TB \otimes \mathcal{L}^{-n}$. We apply these observations when we write a global expression for $\mathcal{O}_\omega$.

Chirality and cohomology of $\mathcal{O}_\omega$. Let us now check that $\mathcal{O}_\omega$ is chiral — annihilated by $\overline{Q}_{\dot\alpha}$ — and moreover represents a nontrivial $\overline{Q}_{\dot\alpha}$ cohomology class. This check follows directly from symmetries.

We recall that the chirality condition on $\mathcal{O}_\omega$ is equivalent to the geometric condition that $\overline{\partial}$ annihilate ω. Because $\mathcal{O}_\omega$ is a pullback from B, we can consider just the action of the $\overline{\partial}$ operator along B on ω, considered as a section of $\overline{\Omega}^p_B \otimes \wedge^p TB \otimes \mathcal{L}^{-n}$. Because both the $\overline{\partial}$ operator on B and ω are singlets under the action of $\mathrm{SU}(2) \times \mathrm{SU}(2n-2)$, the section $\overline{\partial}\omega$ of $\Omega^{p+1}_B \otimes \wedge^p TB \otimes \mathcal{L}^{-n}$ must also be a singlet. But no (nontrivial) invariant section of $\Omega^{p+1}_B \otimes \wedge^p TB \otimes \mathcal{L}^{-n}$ exists; such a section would be constructed from an $\mathrm{SU}(2)$ singlet made from the tensor product of $2p+1$ $\mathbf{2}$'s. So the $\overline{\partial}$ operator on B necessarily annihilates ω.

A similar argument based upon symmetries also shows that $\mathcal{O}_\omega$ cannot be written in the form $\{\overline{Q}_{\dot\alpha}, \mathcal{O}_\xi{}^{\dot\alpha}\}$ in a way that respects the flavor symmetry. Indeed, invariant sections of $\overline{\Omega}^{p-1}_B \otimes \wedge^p TB \otimes \mathcal{L}^{-n}$ and $\overline{\Omega}^p_B \otimes \wedge^{p-1} TB \otimes \mathcal{L}^{-n}$ do not exist, since one cannot make an $\mathrm{SU}(2)$ invariant from $2p-1$ $\mathbf{2}$'s.

[3] There is a further inessential subtlety. A section of TB cannot quite be pulled back to a section of $T\mathcal{M}$ as there is a nontrivial exact sequence $0 \to TF \to T\mathcal{M} \to TB \to 0$ where TF is the tangent space to the fibers of $\mathcal{M} \to B$. Because the relevant cohomology of TF is trivial, this distinction is unimportant.

Global analysis. Our expression in (3.9) is only a local expression for $\mathcal{O}_\omega$, but because the SU($2n$) flavor symmetry acts transitively on $\mathcal{M}$, this local expression suffices to determine a global expression for $\mathcal{O}_\omega$. In order to write such an expression using the mesons M^{ij}, we observe that the local tensors $\epsilon^{s_1 t_1 \cdots s_p t_p}$ and ϵ_{cd} in (3.9) extend globally to tensors on $\mathcal{M}$ given by $\epsilon^{i_1 j_1 \cdots i_n j_n}\,\overline{M}_{i_1 j_1}$ and M^{kl}. Then $\mathcal{O}_\omega$ must take the global form

$$\mathcal{O}_\omega = \Lambda^{6-n}\, F(\overline{M}M)\, \epsilon^{i_1 j_1 \cdots i_n j_n}\, \overline{M}_{i_1 j_1}\, \mathcal{O}_{i_2 j_2} \cdots \mathcal{O}_{i_n j_n}\,, \tag{3.10}$$

with

$$\mathcal{O}_{ij} \,\equiv\, M^{kl}\, \overline{D}\,\overline{M}_{ik} \cdot \overline{D}\,\overline{M}_{lj}\,, \qquad \overline{M}M \,\equiv\, \frac{1}{2}\,\overline{M}_{ij} M^{ij}\,. \tag{3.11}$$

Of course, we employ the usual summation convention in writing $\overline{M}M$ as in (3.11), using the Kähler metric g on $\mathcal{M}$ to raise and lower indices throughout.

In writing $\mathcal{O}_\omega$, we have also included as a prefactor an invariant function $F(\overline{M}M)$ on $\mathcal{M}$ which is not directly determined by the local expression in (3.9). The function $F(\overline{M}M)$ is, however, determined by dimensional analysis and also, as we will now discuss, by requiring $\mathcal{O}_\omega$ to be chiral.

The chirality condition on $\mathcal{O}_\omega$ is most naturally expressed as the condition that the corresponding section ω of $\overline{\Omega}^{n-1}_{\mathcal{M}} \otimes \wedge^{n-1} T\mathcal{M}$ be annihilated by $\overline{\partial}$. Explicitly, the section ω which determines the operator $\mathcal{O}_\omega$ in (3.10) is given globally by

$$\omega = F(\overline{M}M)\, \epsilon^{i_1 j_1 \cdots i_n j_n} \overline{M}_{i_1 j_1} \left(M^{k_2 l_2}\, d\overline{M}_{i_2 k_2} \frac{\partial}{\partial M^{l_2 j_2}} \right) \cdots \left(M^{k_n l_n}\, d\overline{M}_{i_n k_n} \frac{\partial}{\partial M^{l_n j_n}} \right). \tag{3.12}$$

In order that ω be annihilated by $\overline{\partial}$, we have already observed that it must be invariant under the scaling $\overline{M} \to \overline{\lambda}\,\overline{M}$. Furthermore, in order that ω arise from a section of the bundle $\overline{\Omega}^{n-1} \otimes \wedge^{n-1} TB \otimes \mathcal{L}^{-n}$, we have also observed that it must transform under the scaling $M \to \lambda M$ as $\omega \to \lambda^{-n}\omega$.

However, if we ignore $F(\overline{M}M)$, we see that ω in (3.12) otherwise scales with degree n in $\overline{\lambda}$ and with degree zero in λ. Thus, we set $F(\overline{M}M) = (\overline{M}M)^{-n}$ to ensure that ω scales as M^{-n}. So we must set

$$\mathcal{O}_\omega = \Lambda^{6-n}\, \left(\overline{M}M\right)^{-n}\, \epsilon^{i_1 j_1 \cdots i_n j_n}\, \overline{M}_{i_1 j_1}\, \mathcal{O}_{i_2 j_2} \cdots \mathcal{O}_{i_n j_n}\,. \tag{3.13}$$

This expression directly generalizes our previous formula (2.27) in the special case $n = 2$.

Let us also make a remark about the global form of $\mathcal{O}_\omega$, or equivalently ω in (3.12). In this expression, the components M^{ij} of M which appear are just affine coordinates on a vector space in which $\mathcal{M}$ is embedded, and it must be that only the components of $\partial/\partial M^{ij}$ and $d\overline{M}^{ij}$ which represent tangent and cotangent vectors to $\mathcal{M}$ itself appear in (3.12). To check this condition, we can without loss consider the point of $\mathcal{M}$ at which $M^{ij} = \widehat{\epsilon}^{ij}$. (We recall that the nonzero components of $\widehat{\epsilon}$ are $\widehat{\epsilon}^{12} = -\widehat{\epsilon}^{21} = 1$.) Then the holomorphic tangent space to $\mathcal{M}$ at this point is spanned by vectors $\partial/\partial M^{ij}$ for which both $i, j = 1, 2$, corresponding to the singlet Φ, or for which $i = 1, 2$ and $j > 2$, corresponding to Φ^s_c in the representation $(\mathbf{2}, \mathbf{2n-2})$.

In particular, the vector $\partial/\partial M^{ij}$ for which both $i, j > 2$ is not a tangent vector to $\mathcal{M}$ at this point. So in order for (3.12) to be well defined as a section of $\overline{\Omega}^{n-1}_{\mathcal{M}} \otimes \wedge^{n-1} T\mathcal{M}$, such components of $d\overline{M}_{ij}$ and $\partial/\partial M^{ij}$ with both $i, j > 2$ must not appear. However, upon substituting $M^{ij} = \widehat{\epsilon}^{ij}$ into (3.12), we see that the factors of $\overline{M}_{i_1 j_1}$ and M^{kl} ensure that these unwanted components do not appear, and the expression in (3.12) is a section of $\overline{\Omega}^{n-1}_{\mathcal{M}} \otimes \wedge^{n-1} T\mathcal{M}$ as claimed.

Like (2.24), (3.13) is written in terms of an arbitrary unknown Kähler metric on $\mathcal{M}$. As in (2.25), we can make the asymptotic behavior more explicit, since we know the asymptotic form of the Kähler metric. In writing this formula, just as in (2.25), we use Kronecker deltas to raise and lower indices on M (so all components of M and $\overline{M}$ with index up or down have dimension two), and write all factors of $\overline{M}M$ explicitly. With this understood, the asymptotic form of the interaction is

$$\Lambda^{6-n} \left(\overline{M}M\right)^{-(3n-1)/2} \epsilon^{i_1 j_1 \cdots i_n j_n} \, \overline{M}_{i_1 j_1} \, \mathcal{O}_{i_2 j_2} \cdots \mathcal{O}_{i_n j_n} \, . \tag{3.14}$$

4. Computing the multi-fermion F-terms

Although symmetries suffice to fix the form of the F-term correction in SQCD uniquely, we must still check that it is actually generated. So in this section, we provide three computations which show this.

4.1 A direct instanton computation

Since instanton effects are the subject of the paper, we first generate the F-terms directly by a one-instanton computation which generalizes the classic one-instanton computation [1, 4, 5] of the superpotential in the theory with $N_f = N_c - 1$ flavors.

The most basic, and most illuminating, feature of this instanton computation is that it directly explains how the relation $p = n - 1$ arises in the SU(2) theory with $N_f = n$ flavors. This relation arises from counting fermion zero modes in the instanton background, and the same counting implies that, in the SU(N_c) theory, we must have $p = N_f - N_c + 1$.

Very briefly, before we review the details of the instanton computation, we will explain the counting of fermion zero modes that controls the structure of the F-term. We thus recall that, in the one-instanton background, we find at leading order $2N_c$ gaugino zero modes and $2N_f$ quark zero modes. However, beyond leading order, the Yukawa couplings pair $2(N_c - 1)$ of the gaugino and quark zero modes, and these modes are lifted. As a result, two gaugino zero modes and $2(N_f - N_c + 1)$ quark zero modes remain. The two gaugino zero modes that remain are generated by exact global supersymmetries. Thus, if we consider the general form of the multi-fermion F-term in (2.28), the two gaugino zero modes are associated to the fermionic collective coordinates θ^α that appear in the integral over superspace, and the $2(N_f - N_c + 1)$ quark zero modes must be absorbed by the chiral operator $\mathcal{O}_\omega$ itself. So $p = N_f - N_c + 1$.

We now present the details of the instanton computation in the case of SU(2) SQCD. As described above, this computation should generalize directly to the case of SU(N_c) SQCD, though one must consider a more involved integral over the collective coordinates of the instanton.

Following closely the computation of Affleck, Dine, and Seiberg [1], we work on the Higgs branch of SQCD, under the assumption that the classical quark vacuum expectation value, $Q^i_a = v\,\widehat{\delta}^i_a$, is large and the effective gauge coupling $g^2(v)$ is small. In this regime, the approximate instanton equations are valid,

$$D^\mu F_{\mu\nu} = 0\,, \qquad D^2 q^i_a = 0\,, \tag{4.1}$$

where we recall that q^i_a is the scalar component of Q^i_a. In a one-instanton background, the solution of (4.1) for q^i_a with boundary condition fixed by its classical expectation value is given by

$$q^i_a = \frac{\sigma^i_{\mu a}\, x^\mu\, v}{\sqrt{\rho^2 + x^2}}\,. \tag{4.2}$$

Here $\sigma_\mu = (1, -i\sigma^A)$, with σ^A the Pauli matrices, are the usual quaternion representatives. Also, x^μ is a coordinate on $\mathbb{R}^4$, and ρ is the scale of the instanton solution. The classical action for this instanton background is

$$S_0 = \frac{1}{g^2}\left(8\pi^2 + 4\pi^2\rho^2|v|^2\right). \tag{4.3}$$

When $|v|^2 \neq 0$, instantons of large size are exponentially suppressed by this classical action, and the integral over the scale ρ will be convergent.

We must now consider what sort of correlation function to compute in order to probe for the multi-fermion F-term determined by the operator $\mathcal{O}_\omega$ in (3.13). For this purpose, we recall the chiral superfields Φ and Φ^s_c which we introduced in section 3 to describe massless fluctuations of the quark superfields around the Higgs vacuum. Introducing components for these fields,

$$\begin{aligned} \Phi &= \phi + \theta\chi + \cdots\,, \\ \Phi^s_c &= \phi^s_c + \theta\chi^s_c + \cdots\,, \end{aligned} \tag{4.4}$$

we see that among the various interactions which arise from the multi-fermion F-term is an effective interaction for $2n$ fermions of the form

$$\frac{\Lambda^{6-n}}{v^4\,|v|^{2(n-1)}} \int d^4x\, \epsilon^{s_1 t_1 \cdots s_p t_p}\, \epsilon_{c_1 d_1} \cdots \epsilon_{c_p d_p}\, \chi\cdot\chi\, \left(\overline{\chi}^{c_1}_{s_1} \cdot \overline{\chi}^{d_1}_{t_1}\right) \cdots \left(\overline{\chi}^{c_p}_{s_p} \cdot \overline{\chi}^{d_p}_{t_p}\right), \qquad p = n-1\,. \tag{4.5}$$

We have included the dependence of this interaction on v and $\overline{v}$. This dependence can either be checked directly, or it can be deduced from requirement that the interaction transforms as λ^{-n} under $M \to \lambda M$, $\overline{M} \to \overline{\lambda}\,\overline{M}$, as discussed in section 3.

To probe for the presence of the F-term, we thus compute in the instanton background the correlation function[4]

$$\left\langle \overline{\chi}\cdot\overline{\chi} \left(\chi^{s_1}_{c_1} \cdot \chi^{t_1}_{d_1}\right) \cdots \left(\chi^{s_p}_{c_p} \cdot \chi^{t_p}_{d_p}\right) \right\rangle. \tag{4.6}$$

[4]Because the correlator includes external legs with massless propagators, the fermions conjugate to those in the effective vertex appear.

This computation as usual has two pieces: a one-loop integral over fluctuating modes in the instanton background and an integral over zero modes. Because the instanton background is supersymmetric to leading order, the one-loop integral over quantum fluctuations is trivial and contributes only a factor of unity. So the important integral to consider is the integral over zero modes.

Bosonic zero modes. As usual, in the instanton background we have eight bosonic zero modes. Four zero modes are associated to the collective coordinate x_0 for the location of the instanton in $\mathbb{R}^4$. One zero mode is associated to the scale ρ of the instanton. Finally, three zero modes arise from global SU(2) gauge transformations and are associated to a collective coordinate h on SU(2).

Fermionic zero modes. Much more important than the bosonic zero modes are the fermionic zero modes. We have already discussed the counting of these modes generally, but now we review the details.

First, we have two gaugino zero modes which arise from the action of the chiral supercharges Q_α and which take the form

$$\lambda_\alpha^{SS\,A\,[\beta]} = \frac{\rho^2\,\sigma^{A\,\beta}_{\ \ \alpha}}{(\rho^2+x^2)^2}\,. \tag{4.7}$$

Here SS stands for global supersymmetry, A labels the adjoint representation of SU(2), α is a spinor index, and β simply labels the two zero modes. Since we will not try to compute the absolute normalization of our interaction, we have not bothered to normalize the zero modes.

Second, at leading order in g^2, we have an additional $2n+2$ fermion zero modes. Two of these extra zero modes are gaugino zero modes associated to the action of the superconformal generators $x^{\dot\beta}_{\beta} Q^\beta$, of the form

$$\lambda_\alpha^{SC\,A\,[\dot\beta]} = \frac{\rho\,x^{\dot\beta}_{\beta}\,\sigma^{A\,\beta}_{\ \ \alpha}}{(\rho^2+x^2)^2}\,. \tag{4.8}$$

The other $2n$ zero modes arise from the $2n$ fermion doublets and are of the form

$$\psi^i_{\alpha a[j]} = \frac{\rho\,\delta^i_j\,h^b_a\,\epsilon_{\alpha b}}{(\rho^2+x^2)^{3/2}}\,. \tag{4.9}$$

Again, j is just an index that labels the zero modes. We have also included explicitly the dependence of these modes on the element h^b_a of SU(2) parametrizing global gauge transformations. We could also have included this collective coordinate in (4.7) and (4.8), but any dependence of the gaugino zero modes on h will drop out immediately in our computation.

These $2n$ zero modes transform in the representation $\mathbf{2n}$ of the flavor group SU($2n$). After giving expectations to the quark superfields, SU($2n$) is broken to SU(2) $\times$ SU($2n-2$) (where in an instanton field, SU(2) must be combined with a rotation). Under the subgroup, the zero modes of ψ transform as $(\mathbf{2},\mathbf{1})\oplus(\mathbf{1},\mathbf{2n-2})$. The superconformal zero modes similarly transform as $(\mathbf{2},\mathbf{1})$.

Yukawa interactions. The zero modes in (4.7), (4.8), and (4.9) are simply zero modes of the $\not{D}$ operator in the instanton background. However, to perform the instanton computation, we must go beyond leading order and consider the effect of the Yukawa couplings in SQCD. These couplings of course take the form

$$\int d^4x\, \overline{q}_i^a \left(\psi_b^i \cdot \lambda_a^b\right) . \tag{4.10}$$

On the Higgs branch, with q satisfying (4.2), this interaction pairs the two superconformal zero modes λ^{SC} with the two zero modes of the quarks that transform the same way, which are those with $i = 1, 2$ in (4.9) (and which we have denoted χ in (4.4)). As a result, when we compute the correlator (4.6), these fermion zero-modes can be absorbed by pulling down two copies of the Yukawa interaction (4.10) from the SQCD action, which contributes a factor proportional to $\overline{v}^2$ to the correlator.

We are then left with the two gaugino zero modes λ^{SS} and the other $2n-2$ quark zero modes appearing in (4.9). Of course, these $2n-2$ quark zero modes are absorbed directly by the massless fermions χ_c^s appearing in the correlator (4.6). But what of the zero modes λ^{SS}?

To answer this question, we recall that another very important, qualitative effect of the Yukawa coupling (4.10) is that it alters the form of the zero modes λ^{SS} to include components also involving the fermion $\overline{\chi}$. Specifically, to first order in ρv, the relevant equations of motion are

$$\not{D}\,\lambda = 0\,, \qquad \overline{\not{D}}\,\overline{\psi} = \sqrt{2}\,\overline{q}\cdot\lambda\,, \tag{4.11}$$

which have solution

$$\lambda = \lambda^{SS}\,, \qquad \overline{\psi}^{i\,[\beta]}_{\dot\alpha\, a} = \frac{1}{4\pi}\,\overline{\not{D}}^{[\beta]}_{\dot\alpha}\,\overline{q}^i_a\,, \tag{4.12}$$

with $\overline{q}$ as in (4.2). Simply by symmetry, the massless components of $\overline{\psi}^{i\,[\beta]}_{\dot\alpha\, a}$ which mix with λ^{SS} must correspond to the singlet $\overline{\chi}$. Thus, the two supersymmetric zero modes λ^{SS} are absorbed by the two fermions $\overline{\chi}$ which appear in the correlator (4.6).

The classical wavefunction of $\overline{\chi}$ can be explicitly evaluated in the instanton background from (4.12), and far from the instanton location x_0 the wavefunction takes the form

$$\overline{\chi}^{[\beta]}_{\dot\alpha}(x) = \overline{v}\rho^2 S^{\beta}_{\dot\alpha}(x, x_0)\,, \tag{4.13}$$

where $S^{\beta}_{\dot\alpha}(x, x_0)$ is the free fermion propagator.

Computing the correlator. We are now prepared to compute the fermion correlator (4.6) in the instanton background. Using the classical wavefunctions (4.9) and (4.13) for the fermion zero modes, we see that

$$\begin{aligned}\langle \overline{\chi}\cdot\overline{\chi}(\chi^{s_1}_{c_1}\cdot\chi^{t_1}_{d_1})\cdots(\chi^{s_p}_{c_p}\cdot\chi^{t_p}_{d_p})\rangle = \overline{v}^4\,\Lambda^{6-n}\int d^4x_0\, d\rho\, d\mu\, \rho^{2n+5}\exp\left(-\frac{4\pi^2\rho^2|v|^2}{g^2}\right)\times\\ \times\,\epsilon^{s_1t_1\cdots s_pt_p}(h^{e_1}_{c_1}h^{f_1}_{d_1}\,\epsilon_{e_1f_1})\cdots(h^{e_p}_{c_p}h^{f_p}_{d_p}\,\epsilon_{e_pf_p})\times\\ \times\,(S(y_1-x_0)\cdot S(y_2-x_0))\cdots\times\\ \times\,(S(y_{2n-1}-x_0)\cdot S(y_{2n}-x_0))\,.\end{aligned} \tag{4.14}$$

In this expression, $y_1, \ldots, y_{2n}$ are the positions of the $2n$ fermions in $\mathbb{R}^4$, which are assumed to be far from the position x_0 of the instanton. We then make use of the fact that, in this limit, the classical wavefunctions (4.9) of the fermions χ^s_c have the correct asymptotic behavior so that the correlator can be written using the free fermion propagator S. In computing the amputated vertex, we would simply drop these factors and the integration over the position x_0 of the instanton.

Besides the factor d^4x_0, the bosonic measure also includes a factor $d\rho\,\rho^{2n+5}$ and a factor $d\mu$, which represents the invariant Haar measure on SU(2). We have determined the power of ρ that appears simply by dimensional analysis.

Thus, since a prefactor of $\overline{v}^4$ appears from the fermion zero modes, the gaussian integral over ρ then produces the correct dependence on v and $\overline{v}$ as in (4.5). We have not been careful about factors of the gauge coupling g^2 which also appear in the integration measure and upon performing the gaussian integral. By holomorphy, any explicit dependence of the correlator on g^2 should be absorbed into a wavefunction renormalization of the external legs.

The only integral left to consider is the group integral over SU(2), which takes the form

$$I^{d_1\,d_2\cdots d_{2p}}_{c_1\,c_2\cdots c_{2p}} = \int d\mu\, h^{d_1}_{c_1}\, h^{d_2}_{c_2} \cdots h^{d_{2p}}_{c_{2p}}\,. \tag{4.15}$$

This integral is not identically zero, since if one picks a real unit vector n^c_d and contracts with $n^{c_1}_{d_1} n^{c_2}_{d_2} \ldots n^{c_{2p}}_{d_{2p}}$, the integrand on the right hand side becomes positive definite. The $\mathrm{SU}(2) \times \mathrm{SU}(2)$ symmetry implies that

$$I^{d_1\,d_2\cdots d_{2p}}_{c_1\,c_2\cdots c_{2p}} \propto \epsilon^{d_1 d_2}\,\epsilon_{c_1 c_2} \cdots \epsilon^{d_{2p-1} d_{2p}}\,\epsilon_{c_{2p-1} c_{2p}} + (\text{permutations})\,. \tag{4.16}$$

Here the first term on the right hand side must be symmetrized under the exchanges of indices corresponding to exchanges between the factors of h in (4.15). (Note that, by bose statistics, a term proportional to $\epsilon^{d_i d_j}$ for some i and j is also proportional to $\epsilon_{c_i c_j}$.) These symmetries arise in the effective interaction (4.5) from the permutation symmetries of the fermions. Thus, upon substituting (4.16) into (4.14), we produce the effective interaction which arises from the multi-fermion F-term.

4.2 A computation in the Seiberg dual with six doublets

In many examples of duality, nonperturbative effects in the direct theory become classical effects in the dual theory. An example is the SU(2) gauge theory with $2n = 4$, that is with four doublets. In this case, the basic nonperturbative effect, as we reviewed in section 2, is the deformation of the moduli space $\mathcal{M}$ of vacua. The dual theory that describes the infrared physics is the sigma model whose target is $\mathcal{M}$, and the complex structure is built in at tree level.

Here we want to describe a more subtle example of this, for the SU(2) theory with $2n = 6$ doublets. In this case we will show that the multi-fermion F-term that we have obtained nonperturbatively in the direct description of SQCD can also be computed at tree level in the Seiberg dual [6, 8] description.

As promised in section 2, we also reconsider here the deformation of complex structure that occurs in the theory with four doublets. In particular, we reproduce the effective interaction in (2.27) by integrating out the massive fields in the linear sigma model with superpotential $W = \Sigma(M \wedge M - \Lambda^4)$ which describes the deformation. Since this computation is exactly the same in spirit as our classical computation in the Seiberg dual of the theory with six doublets, we describe both computations together.

The Seiberg dual of SU(2) SQCD with six doublets is distinguished by the fact that the dual gauge group is trivial, and hence this theory is especially simple. In particular, the elementary degrees of freedom in the dual theory are described entirely by the mesonic fields M^{ij}, with Wess-Zumino action

$$S = \frac{1}{\mu^2} \int d^4x\, d^4\theta\, \overline{M} M + \int d^4x\, d^2\theta\, \Lambda^{-3}\, M \wedge M \wedge M + \text{c.c.} \tag{4.17}$$

We have included the canonical kinetic terms in S, with an arbitrary scale μ that appears so that, by convention, M has dimension two. Using a different kinetic term for M would not affect the computation of F-terms.

The cubic superpotential plays an interesting role in this theory. As shown by Seiberg [6], this potential appears nonperturbatively in the electric theory, but in the dual theory it arises at tree level. In either case, the F-term equations which follow from this superpotential are simply the classical Plücker relations $M \wedge M = 0$ that enforce the condition $\text{rank}(M) \leq 2$, which is necessary to describe $\mathcal{M}$.

In the special case $n = 3$, the multi-fermion F-term in (3.13) takes the explicit form

$$\begin{aligned} \delta S = \frac{1}{\mu^4} \int d^4x\, d^2\theta\, \Lambda^3\, (\overline{M} M)^{-3}\, \epsilon^{i_1 j_1 i_2 j_2 i_3 j_3}\, \overline{M}_{i_1 j_1} \times \\ \times \left(M^{kl}\, \overline{D}\,\overline{M}_{i_2 k} \cdot \overline{D}\,\overline{M}_{l j_2}\right) \left(M^{k' l'}\, \overline{D}\,\overline{M}_{i_3 k'} \cdot \overline{D}\,\overline{M}_{l' j_3}\right). \end{aligned} \tag{4.18}$$

We will generate this effective interaction in the most naive way possible. We simply observe that, when we expand the Wess-Zumino model around a generic point on $\mathcal{M}$, the cubic superpotential induces a mass for some components of M. We then integrate out these massive modes at tree level in a Feynman diagram computation to generate (4.18).

At this point, one might immediately protest that we are making the quixotic proposal to generate an F-term in perturbation theory and in blatant violation of standard non-renormalization theorems. However, these non-renormalization theorems have only been considered for conventional F-terms which describe superpotentials, and the multi-fermion F-terms we consider evade them in an interesting way.

The essential point here is that the multi-fermion F-terms arise from cohomology classes on $\mathcal{M}$. Whenever we perform a perturbative computation around some vacuum on $\mathcal{M}$, we are only working in a small neighborhood of that point, and in that neighborhood any operator $\mathcal{O}_\omega$ which represents a positive degree cohomology class of $\overline{Q}_{\dot\alpha}$ becomes $\overline{Q}_{\dot\alpha}$-trivial. As a result, though globally on $\mathcal{M}$ the multi-fermion F-terms cannot be written as D-terms, they can be written as D-terms if we expand in fluctuations around a given vacuum. These D-terms can then be directly generated in perturbation theory.

As a simple and highly relevant example, we consider the F-term at hand in (4.18). We expand (4.18) around some point with $\langle M^{ij}\rangle \neq 0$. With no loss of generality, we can assume that the only nonzero component of $\langle M^{ij}\rangle$ is $\langle M^{12}\rangle$. In expanding around this particular vacuum, we apply our standard convention that c, d, e, f refer to indices $1, 2$, s, t, u, v refer to indices $3, \ldots, 6$, and i, j, k, l run over all indices $1, \ldots, 6$. From (4.18), we generate a series of interactions among the fluctuating fields δM, one interaction being

$$\delta S = \frac{1}{\mu^4}\int d^4x\, d^2\theta\, \Lambda^3\, \langle \overline{M}M\rangle^{-3}\langle \overline{M}_{12}\rangle \times \\ \times\, \epsilon^{s_1 t_1 s_2 t_2}(\delta M^{cd}\, \overline{D}\, \delta\overline{M}_{s_1 c}\cdot \overline{D}\, \delta\overline{M}_{dt_1})(\delta M^{ef}\, \overline{D}\, \delta\overline{M}_{s_2 e}\cdot \overline{D}\, \delta\overline{M}_{ft_2})\,. \quad (4.19)$$

Of course, the effective fermion interaction (4.5) which we considered in the instanton computation is one of the terms that arises from (4.19).

By definition, if $\delta\overline{M}_{ij}$ is massless, then the basic equation of motion (2.36) for $\delta\overline{M}_{ij}$ takes the form $\overline{D}^2\, \delta\overline{M}_{ij} = \mathcal{O}(\delta M^2)$. Since only massless fluctuations appear in the effective interaction (4.19), we can immediately integrate this F-term into a D-term at leading order,

$$\delta S = \frac{1}{\mu^4}\int d^4x\, d^4\theta\, \Lambda^3 \langle \overline{M}M\rangle^{-3}\langle \overline{M}_{12}\rangle \times \\ \times\, \epsilon^{s_1 t_1 s_2 t_2}(\delta M^{cd}\, \delta\overline{M}_{s_1 c}\, \delta\overline{M}_{dt_1})(\delta M^{ef}\, \overline{D}\, \delta\overline{M}_{s_2 e}\cdot \overline{D}\, \delta\overline{M}_{ft_2})\,. \quad (4.20)$$

We have used the fact that to this order, two $\overline{D}$'s cannot act on the same $\delta\overline{M}$, and none can act on δM.

In the case of the theory with $n = 2$, the same observations imply that the analogous part of the F-term in (2.27) can be rewritten locally as the simple D-term below,

$$\delta S = \frac{1}{\mu^2}\int d^4x\, d^4\theta\, \Lambda^4\, \langle \overline{M}M\rangle^{-1}\, \epsilon^{cd}\epsilon^{st}\, \delta\overline{M}_{sc}\delta\overline{M}_{dt}\,. \quad (4.21)$$

Here again we expand around a vacuum in which the nonvanishing part of $\langle M\rangle$ is $\langle M^{12}\rangle$, and $c, d = 1, 2$ while $s, t = 3, 4$.

Thus, the appearance of these unusual F-terms is signaled by the perturbative appearance of the D-terms in (4.20) and (4.21), which we must now compute. As in the instanton computation, we could compute some particular component of this superspace interaction. However, we are in a situation perfectly suited for a manifestly supersymmetric computation using the formalism of super Feynman diagrams.

Evaluating a super Feynman diagram. We will not review here the basic derivation of Feynman rules in superspace, for which we recommend section 6.3 of [22]. In general, superspace Feynman rules can be derived by standard path integral manipulations just as for ordinary Feynman rules, and for the sake of brevity we will only state the super Feynman rules that we need for our very simple, tree-level computations.

In the case of the theory with $2n = 6$, we begin by expanding the tree-level Wess-Zumino action in fluctuations δM about the vacuum, so that

$$S = \frac{1}{\mu^2}\int d^4x\, d^4\theta\, \delta\overline{M}\delta M + \int d^4x\, d^2\theta\, (3\,\lambda\, \langle M\rangle\wedge\delta M\wedge\delta M + \lambda\, \delta M\wedge\delta M\wedge\delta M) + \text{c.c.}\,, \quad (4.22)$$

where for convenience we introduce the abbreviation

$$\lambda \equiv \Lambda^{-3}\,. \tag{4.23}$$

We will not be concerned with constants here, and we simply absorb the numerical factor of 3 in (4.22) into $\langle M\rangle$. We will also suppress the appearance of the mass scale μ in all expressions that follow, since its appearance is trivially fixed at the end of the computation by dimensional analysis.

Of course, we similarly expand the sigma model action in the theory with $n = 2$,

$$S = \frac{1}{\mu^2}\int d^4x\, d^4\theta\, \delta\overline{M}\delta M + \int d^4x\, d^2\theta\, (2\,\langle M\rangle\wedge\delta M\,\delta\Sigma + \delta M\wedge\delta M\,\delta\Sigma - \varepsilon\,\delta\Sigma) + \text{c.c.} + \cdots\,, \tag{4.24}$$

where the ellipses indicate kinetic terms and a mass term for the fluctuations of the auxiliary field Σ. As above, we ignore constants, and we abbreviate

$$\varepsilon \equiv \Lambda^4\,. \tag{4.25}$$

The most important terms in (4.24) for our computation are simply the linear source term for $\delta\Sigma$ which represents the deformation as well as the mass term mixing δM and $\delta\Sigma$.

Propagators. In the vacuum with only $\langle M^{12}\rangle \neq 0$, we want to get an effective interaction for the massless fields by integrating out the massive fields M^{st}, $s, t = 3, \ldots, 2n$.

These fields have standard superspace propagators, which may be either chiral or non-chiral. We indicate these propagators below, in the theory with $n = 3$,

$$\begin{aligned}
\delta\overline{M}_{st}\text{———————}\;\delta M^{uv} &= \frac{\delta^{uv}_{st}}{(p^2 + \overline{\lambda}\lambda\langle\overline{M}M\rangle)}\,,\\
\delta M^{st}\;\overset{D^2}{-\,-\,-\,-\,-\,-}\;\delta M^{uv} &= \frac{\overline{\lambda}\,\langle\overline{M}_{12}\rangle\,\epsilon^{stuv}\,D^2}{p^2(p^2 + \overline{\lambda}\lambda\langle\overline{M}M\rangle)}\,,\\
\delta\overline{M}_{st}\;\overset{\overline{D}^2}{-\,-\,-\,-\,-\,-}\;\delta\overline{M}_{uv} &= \frac{\lambda\,\langle M^{12}\rangle\,\epsilon_{stuv}\,\overline{D}^2}{p^2(p^2 + \overline{\lambda}\lambda\langle\overline{M}M\rangle)}\,.
\end{aligned} \tag{4.26}$$

In writing the non-chiral propagator, we use the standard notation $\delta^{uv}_{st} = \delta^u_s\delta^v_t - \delta^u_t\delta^v_s$. We have also suppressed a superspace delta function $\delta^4(\theta-\theta')$ which accompanies these propagators. Finally, we note the superspace derivatives D^2 and $\overline{D}^2$ which appear in the chiral and anti-chiral propagators. These factors arise ubiquitously in supergraph computations when chiral integrals over half of superspace are rewritten as non-chiral integrals over the full superspace.

In the theory with $n = 2$, similar propagators appear for the appropriate linear combinations of $\delta\Sigma$ and δM, for which the mass squared is again proportional[5] to $\langle\overline{M}M\rangle$.

[5] If a separate mass term $m\Sigma^2$ for Σ is also present, this statement remains true in the classical limit that $\langle\overline{M}M\rangle$ is large.

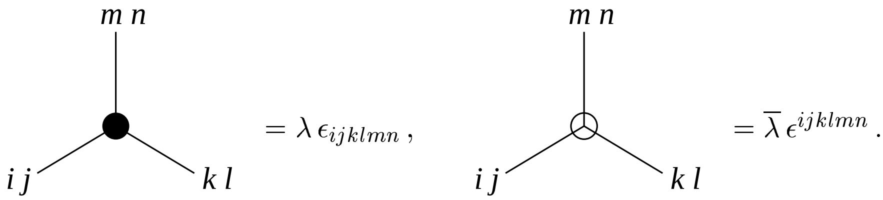

Figure 1: Vertices for $n = 3$.

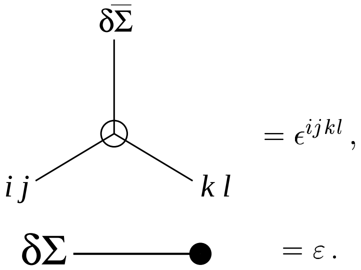

Figure 2: Vertices for $n = 2$.

Vertices. In the theory with $n = 3$, the cubic superpotential gives rise to cubic vertices for chiral and anti-chiral interactions, as we distinguish in figure 1. We have written these interactions in an SU(6) symmetric fashion, though of course each chiral and anti-chiral vertex decomposes under the unbroken $\mathrm{SU}(2) \times \mathrm{SU}(4)$ symmetry to give various interactions between the massive and massless components of M, which we leave implicit. Each superspace vertex comes with a factor of $\int d^4\theta$, and the delta functions from the propagators simply ensure that the overall diagram has precisely one factor of $\int d^4\theta$, as we expect.

In the corresponding theory with $n = 2$, we require a similar cubic vertex arising from the interaction $\delta\overline{M}\wedge\delta\overline{M}\,\delta\overline{\Sigma}$ as well as the chiral source term $\varepsilon\,\delta\Sigma$, as shown in figure 2. Again, we leave the obvious decomposition under $\mathrm{SU}(2) \times \mathrm{SU}(2)$ implicit.

Last, we recall the rule that if a chiral vertex has N internal legs (external legs don't count), then $N-1$ of those legs appear with a factor of $\overline{D}^2$ attached. Briefly, if $J(x,\theta)$ is the chiral source introduced as usual to derive Feynman rules, then the functional derivative of J satisfies $\delta J(x,\theta)/\delta J(x',\theta') = \overline{D}^2\,\delta^4(x-x')\,\delta^4(\theta-\theta')$. So N factors of $\overline{D}^2$ appear from these derivatives, but one factor of $\overline{D}^2$ is used to write $\int d^2\theta\,\overline{D}^2 = \int d^4\theta$, as mentioned above.

With these rules in hand, we can immediately generate the interactions in (4.20) and (4.21). First, in the simpler case of $n = 2$, we immediately evaluate the simple diagram in figure 3 at zero momentum to produce the effective interaction

$$\int d^4x\, d^4\theta\, \epsilon^{cd}\epsilon^{st}\,\delta\overline{M}_{sc}\delta\overline{M}_{dt}\,\frac{\varepsilon}{\langle\overline{M}M\rangle}\,, \tag{4.27}$$

as in (4.21).

For the theory with $n = 3$, we consider the slightly more involved diagram in figure 4. We note that the D^2 operator in this diagram arises from the central chiral propagator, and the two $\overline{D}^2$ operators arise from the two chiral vertices.

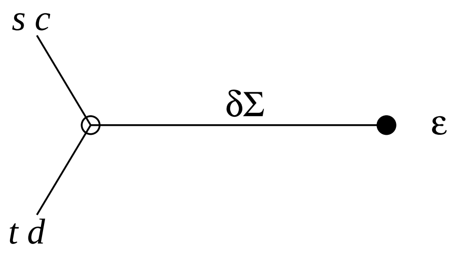

Figure 3: Two-point super Feynman diagram.

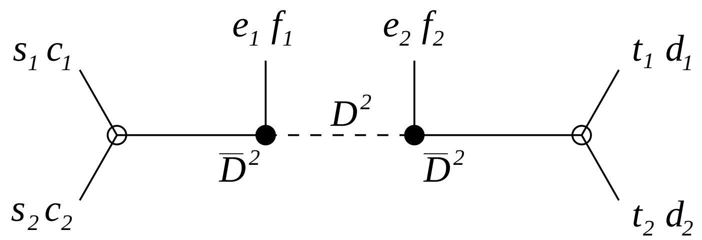

Figure 4: Six-point super Feynman diagram.

At first sight, one might worry about the spurious pole at zero momentum that appears to arise from the extra factor of p^2 appearing in the central chiral propagator, as in (4.26). Physically, since we only integrate out massive fields, we do not expect to find any pole at zero momentum.

However, we can integrate by parts to move one of the $\overline{D}^2$ operators onto the central chiral propagator to form $\overline{D}^2 D^2$. Since $\overline{D}^2 D^2 = p^2$ when acting on a chiral field, this factor of $\overline{D}^2 D^2$ cancels against the extra factor of p^2 in the denominator of the chiral propagator. Thus, the diagram is well defined in the limit of zero momentum, and we evaluate it in this limit to reproduce the D-term (4.20). We also note that once we cancel the factor of $\overline{D}^2 D^2$, we are left with only one factor of $\overline{D}^2$, which acts on the external anti-chiral legs just as in the interaction (4.20).

So at zero momentum, the remainder of our computation is a trivial matter of algebra. We find that this diagram produces the effective interaction

$$\begin{aligned} &\int d^4x\, d^4\theta\, \delta\overline{M}_{s_1c_1}\, \delta\overline{M}_{s_2c_2}\, \left(\overline{D}\delta\overline{M}_{t_1d_1} \cdot \overline{D}\delta\overline{M}_{t_2d_2}\right)\, \delta M^{e_1f_1}\, \delta M^{e_2f_2} \times \\ &\times \frac{\overline{\lambda}\, \epsilon^{c_1c_2}\, \epsilon^{s_1s_2uv}}{\overline{\lambda}\lambda\, \langle\overline{M}M\rangle} \cdot \lambda\, \epsilon_{uvu'v'}\, \epsilon_{e_1f_1} \cdot \frac{\overline{\lambda}\, \epsilon^{u'v'wx}\, \langle\overline{M}_{12}\rangle}{\overline{\lambda}\lambda\, \langle\overline{M}M\rangle} \cdot \lambda\, \epsilon_{wxw'x'}\, \epsilon_{e_2f_2} \cdot \frac{\overline{\lambda}\, \epsilon^{w'x't_1t_2}\, \epsilon^{d_1d_2}}{\overline{\lambda}\lambda\, \langle\overline{M}M\rangle}\,. \end{aligned} \tag{4.28}$$

The tensor on the second line of (4.28) is then proportional to

$$\lambda^{-1}\, \langle\overline{M}M\rangle^{-3}\, \langle\overline{M}_{12}\rangle\, \delta^{c_1c_2}_{e_1f_1}\, \delta^{d_1d_2}_{e_2f_2}\, \epsilon^{s_1s_2t_1t_2}\,, \tag{4.29}$$

which has precisely the form required to produce the F-term. The $\overline{\lambda}$'s have happily canceled, ensuring the requisite holomorphy.

N. Seiberg pointed out the following interpretation of the $1/\lambda$ factor. As the meson superfield M has dimension two in the classical theory, the superpotential interaction proportional to $\int d^4x\, d^2\theta\, M \wedge M \wedge M$ must on dimensional grounds be interpreted in the underlying SQCD theory as $\int d^4x\, d^2\theta\, \Lambda^{-3}\, M \wedge M \wedge M$. Thus, what we have called λ is a multiple of Λ^{-3} in SQCD, as in (4.23). Hence the multi-fermion F-term interaction, being proportional to λ^{-1} in the Seiberg dual description, is proportional to Λ^3 in the original SQCD description. Λ^3 is the standard instanton factor for SU(2) with six doublets, and the direct instanton computation of section 4.1 did, accordingly, give a result proportional to Λ^3.

4.3 Mass deformation and renormalization group flow

For our final computation, we perturb SU(2) SQCD with $2n$ massless doublets by adding a tree-level superpotential which gives a mass to some of the n flavors,

$$W = m_{ij}\, M^{ij}\,. \tag{4.30}$$

As usual, we assign charges to the mass parameters m_{ij} under the symmetries of the massless theory so that W is formally invariant,

$$\begin{array}{ccccc} & \mathrm{SU}(2) & \mathrm{SU}(2n) & \mathrm{U}(1)_A & \mathrm{U}(1)_{\mathcal{R}} \\ m_{ij} & \mathbf{1} & \wedge^2(\overline{\mathbf{2n}}) & -2 & -2(1-\frac{2}{n})\,. \end{array} \tag{4.31}$$

The whole computation will be performed on B.

As we observed in general in section 2.3, the tree-level superpotential alters the on-shell supersymmetry algebra of the theory. Consequently, the operator $\mathcal{O}_\omega \equiv \mathcal{O}_\omega^{(n)}$ in (3.13) which is chiral in SQCD with $2n$ massless doublets is no longer chiral when some of those doublets become massive.

Physically, we expect that there is instead some deformation $\widetilde{\mathcal{O}}_\omega$ of this operator, depending holomorphically on m_{ij}, which is chiral in the massive theory and which reduces to $\mathcal{O}_\omega^{(n)}$ upon setting m_{ij} to zero.

On the other hand, if we give very large masses to k of the flavors and integrate them out, we also expect that $\widetilde{\mathcal{O}}_\omega$ must reduce to the operator $\mathcal{O}_\omega^{(n-k)}$ appropriate for the massless theory with $n-k$ flavors. In particular, upon integrating out all but one flavor, $\widetilde{\mathcal{O}}_\omega$ should reproduce the well known nonperturbative superpotential,

$$W = \frac{\Lambda^5}{M}\,. \tag{4.32}$$

Here $M = M^{12}$ is the only independent component of the 2×2 antisymmetric matrix M^{ij}.

We now compute $\widetilde{\mathcal{O}}_\omega$, which will be uniquely determined from $\mathcal{O}_\omega^{(n)}$ by supersymmetry and will have the properties above. Since we know already from the work of [1] that the superpotential (4.32) is generated, we will thus show that the F-term involving $\mathcal{O}_\omega^{(n)}$ is generated in the massless theory with n flavors. Finally, we remark that this sort of analysis extends, at least in spirit, directly to the general case of SU(N_c) SQCD with $N_f > N_c$ flavors and might be successfully applied there.

As before, we use $\mathcal{M}$ to denote the moduli space of the massless theory, and we recall that $\mathcal{M}$ is a complex cone over the Grassmannian $B = \mathrm{SU}(2n)/S(\mathrm{U}(2)\times\mathrm{U}(2n-2))$. Then

JHEP01(2005)056

our problem of constructing $\widetilde{\mathcal{O}}_\omega$ is equivalent to the geometric problem of finding a tensor $\widetilde{\omega}$, which is generally an inhomogeneous sum of sections of $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$ for various p, such that $\widetilde{\omega}$ satisfies the supersymmetry condition,

$$\left(\overline{\partial} + \iota_{dW}\right) \widetilde{\omega} = 0 \,, \tag{4.33}$$

and in the massless limit reduces to our former tensor ω.

Preliminaries. As in section 3, the important analysis of $\widetilde{\omega}$ is the local analysis on B near the point corresponding to $M^{ij} = \widehat{\epsilon}^{ij}$. However, we first find it useful to revisit our construction of the simpler tensor ω in greater detail and in a manner which immediately generalizes to the construction of $\widetilde{\omega}$.

Let us recall our construction of ω in section 3. We begin by picking a point P on B, for concreteness corresponding to the point $M^{ij} = \widehat{\epsilon}^{ij}$ on $\mathcal{M}$. Since the M^{ij} are homogeneous coordinates on B, we can use the scaling symmetry to set $M^{12} = 1$. Then we take complex coordinates on B (in a neighborhood of the point P) to be simply the off-diagonal matrix elements $\phi^s_c = \epsilon_{cd} M^{sd}$, $c = 1, 2$, $s = 3, \ldots, 2n$. They transform as in (3.5) under the action of the unbroken $S(\mathrm{U}(2) \times \mathrm{U}(2n-2))$ symmetry group at P. The matrix elements M^{ij}, $i, j > 2$, are determined in terms of the ϕ^s_c by the equation $M \wedge M = 0$. We will not need the explicit form of these matrix elements; the only important fact is that they are of order $(\phi^s_c)^2$.

In (3.13), we determined the form of the multi-fermion F-term:

$$\mathcal{O}_\omega = \Lambda^{6-n} \, \left(\overline{M} M\right)^{-n} \, \epsilon^{i_1 j_1 \cdots i_n j_n} \, \overline{M}_{i_1 j_1} \, \mathcal{O}_{i_2 j_2} \cdots \mathcal{O}_{i_n j_n} \,, \tag{4.34}$$

where

$$\mathcal{O}_{ij} \;\equiv\; M^{kl} \, \overline{D} \, \overline{M}_{ik} \cdot \overline{D} \, \overline{M}_{lj} \,, \qquad \overline{M} M = \frac{1}{2} \overline{M}_{ij} M^{ij} \,. \tag{4.35}$$

In that discussion, we used an argument based on symmetries to prove that $\overline{\partial}\omega = 0$. As a prelude to including the superpotential deformation, we will here demonstrate this more explicitly.

Since ω is invariant under the action of $\mathrm{SU}(2n)$ on the homogeneous space B, it suffices to show that $\overline{\partial}\omega = 0$ at the point P. As $\overline{\partial}$ is a first order differential operator, to evaluate $\overline{\partial}\omega$ at P, it suffices to describe ω near P to within terms of order ϕ^2. This leads to drastic simplification. For example, $\overline{M} M = 1 + \mathcal{O}(\phi^2)$, so we can simply set $\overline{M} M = 1$. Furthermore, examination of (4.34) and (4.35) shows that up to terms of order ϕ^2, we can replace the explicit factor of $\overline{M}_{i_1 j_1}$ in (4.34) by $\widehat{\epsilon}_{i_1 j_1}$, so that all factors of $\mathcal{O}_{i_k j_k}$ have $i_k, j_k > 2$. Further, for $i_k, j_k > 2$, we can take $\mathcal{O}_{i_k j_k} = \epsilon^{cd} \overline{D} \, \overline{M}_{i_k c} \cdot \overline{D} \, \overline{M}_{d j_k}$, again up to terms of order ϕ^2. The net effect is that, up ϕ^2 terms,

$$\omega = \epsilon^{s_1 t_1 s_2 t_2 \ldots s_n t_n} \left(\epsilon_{c_1 d_1} d\overline{\phi}^{c_1}_{s_1} \frac{\partial}{\partial \phi^{t_1}_{d_1}} \right) \cdots \left(\epsilon_{c_n d_n} d\overline{\phi}^{c_n}_{s_n} \frac{\partial}{\partial \phi^{t_n}_{d_n}} \right) . \tag{4.36}$$

Now the fact that $\overline{\partial}\omega = 0$ at P is completely obvious: all terms in ω have constant coefficients (up to terms of order ϕ^2 that have been dropped), so there is nothing that the derivatives in $\overline{\partial}$ can act on.

The benefit of this approach is that we can now conveniently understand the generalization with the superpotential turned on. We claim that the generalization of $\mathcal{O}_\omega$ is simply

$$\begin{aligned}\widetilde{\mathcal{O}}_\omega &= \Lambda^{6-n}\left(\overline{M}M\right)^{-n}\epsilon^{i_1j_1\cdots i_nj_n}\,\overline{M}_{i_1j_1}\,\widetilde{\mathcal{O}}_{i_2j_2}\cdots\widetilde{\mathcal{O}}_{i_nj_n}\,,\\ \widetilde{\mathcal{O}}_{ij} &\equiv M^{kl}\,\overline{D}\,\overline{M}_{ik}\cdot\overline{D}\,\overline{M}_{lj}-\left(\overline{M}M\right)m_{ij}\,.\end{aligned}\tag{4.37}$$

This certainly reduces to $\mathcal{O}_\omega$ at $m=0$; we just have to prove that it is chiral. In other words, we need to show that the object $\widetilde{\omega}$, obtained from ω by replacing each $\mathcal{O}_{ij}$ by $\widetilde{\mathcal{O}}_{ij}$, is annihilated by $\overline{\partial}+\iota_{dW}$. It suffices to do the computation at the point $P\in B$ with $M^{ij}=\widehat{\epsilon}^{ij}$ since, as we will make no particular assumption about the form of the mass matrix m_{ij}, the computation would proceed in the same way at any other point.

So as before, we want to write out a simple formula for $\widetilde{\omega}$ that is valid near P to within an error of order ϕ^2. The same reasoning applies as before: the explicit factors of $\overline{M}_{i_1j_1}$ in $\widetilde{\mathcal{O}}_\omega$ and of M^{kl} in $\widetilde{\mathcal{O}}_{ij}$ can be replaced by $\widehat{\epsilon}_{i_1j_1}$ and $\widehat{\epsilon}^{kl}$. Since in $\widetilde{\mathcal{O}}_{ij}$, the indices i,j are then in the range $3,\ldots,2n$ the mass matrix m_{ij} can be replaced by μ_{ij}, its orthogonal projection onto the part with $i,j>2$. We write Π for the projector onto components of m with $i,j>2$ and will describe Π more explicitly momentarily.

The upshot is that up to terms of order ϕ^2, $\widetilde{\omega}$ is described near P by a simple generalization of (4.36),

$$\begin{aligned}\widetilde{\omega} &= \epsilon^{s_1t_1\cdots s_pt_p}\,\widetilde{\omega}_{s_1t_1}\cdots\widetilde{\omega}_{s_pt_p}\,,\qquad p=n-1\,,\widetilde{\omega}_{st}\\ &= \epsilon_{cd}\left(d\overline{\phi}^c_s\frac{\partial}{\partial\phi^t_d}+\frac{\partial}{\partial\phi^s_c}d\overline{\phi}^d_t\right)-\mu_{st}\,.\end{aligned}\tag{4.38}$$

The virtue of factorizing $\widetilde{\omega}$ in this way is as we will see each factor $\widetilde{\omega}_{s_it_i}$ is separately annihilated by $\overline{\partial}+\iota_{dW}$. Also, in the expression for $\widetilde{\omega}_{st}$ in the second line of (4.38), we have explicitly indicated the two terms that arise from the contraction of spinor indices on $\overline{D}_{\dot{\alpha}}$ in (4.37), since we will try to be careful about factors of two in the following.

Let us first evaluate $\iota_{dW}(\widetilde{\omega}_{st})$. The contraction operator ι_{dW} trivially annihilates μ_{st} (because the latter is a zero-form). As $W=m_{ij}M^{ij}$, we have $dW=m_{ij}dM^{ij}$. So the effect of contraction with dW is just to map $\partial/\partial\phi^s_c$ to μ^c_s, the projection of the mass matrix m to terms that transform like $\partial/\partial\phi^s_c$ (in other words, as $(\mathbf{2},\overline{\mathbf{2n-2}})$) under the subgroup of the symmetry group that leaves fixed the point $P\in B$. Hence we have

$$\iota_{dW}\widetilde{\omega}_{st}=\epsilon_{cd}\left(d\overline{\phi}^c_s\,\mu^d_t+\mu^c_s\,d\overline{\phi}^d_t\right).\tag{4.39}$$

It remains to evaluate $\overline{\partial}(-\mu_{st})$. This is nonzero because of the projection in the definition of μ_{st}. As we will show,

$$\overline{\partial}\mu_{st}=\epsilon_{cd}\left(d\overline{\phi}^c_s\,\mu^d_t+\mu^c_s\,d\overline{\phi}^d_t\right).\tag{4.40}$$

From (4.39) and (4.40), we then see directly that $\overline{\partial}+\iota_{dW}$ annihilates $\widetilde{\omega}_{st}$ and hence $\widetilde{\omega}$ at the point P on B.

To derive the formula (4.40) for $\overline{\partial}\mu_{st}$, we begin by considering the projection Π of the mass matrix m onto its components which transform in the representation $\wedge^2\left(\overline{\mathbf{2n-2}}\right)$. We can directly write a global formula for this projection,

$$\begin{aligned}\Pi(m)_{ij} &= m_{ij} + (\overline{M}M)^{-1}(m_{ik}M^{kl}\overline{M}_{lj} - m_{jk}M^{kl}\overline{M}_{li}) + \\ &\quad +(\overline{M}M)^{-2}(\overline{M}_{ik}M^{kl}m_{lp}M^{pq}\overline{M}_{qj})\,.\end{aligned} \tag{4.41}$$

Upon substituting $M^{ij} = \widehat{\epsilon}^{ij}$ and using repeatedly that $\widehat{\epsilon}^{kl}\widehat{\epsilon}_{lj} = -\widehat{\delta}^k_j$ (explaining the signs above), one can check that the second and third terms of (4.41) subtract the components of m transforming in the representations $\mathbf{1}$ and $(\mathbf{2},\overline{\mathbf{2n-2}})$ under $\mathrm{SU}(2)\times\mathrm{SU}(2n-2)$ at P, leaving only the components in $\wedge^2\left(\overline{\mathbf{2n-2}}\right)$. Since the formula (4.41) for Π is invariant, it is correct globally on B.

Because the action of $\overline{\partial}$ commutes with pullback, we can now act with $\overline{\partial}$ directly on (4.41) as an unconstrained expression in the ambient vector space (or projective space) parametrized by M^{ij}. We then pull this expression back to $\mathcal{M}$ by dropping all terms which involve the one-forms $d\overline{M}_{ij}$ with both indices $i,j>2$.

To evaluate $\overline{\partial}\mu$ at $\phi=0$, we can discard all terms proportional to ϕ, and in particular to components M^{ij} or $\overline{M}_{ij}$ with i or j bigger than 2. Terms that survive at $\phi=0$ only arise from the action of $\overline{\partial}$ on the second term of (4.41), with the expression

$$\left(\overline{M}M\right)^{-1}\left(m_{ik}M^{kl}d\overline{M}_{lj} - m_{jk}M^{kl}d\overline{M}_{li}\right)\,, \qquad i,j>2\,. \tag{4.42}$$

From this global expression (4.42) we immediately deduce the local formula (4.40) upon setting $M^{ij} = \widehat{\epsilon}^{ij}$ and identifying $m_{ik}M^{kl}d\overline{M}_{lj}$ as representing locally $\epsilon_{cd}\,\mu^c_s\,d\overline{\phi}^d_t$ at P. We remark that the relative sign between the two terms in (4.40) and (4.42) arises from a rearrangement of flavor indices in passing from (4.42) to (4.40).

Finally, although we have thus far only considered the special case that $W = m_{ij}M^{ij}$, if we now consider the case of a general superpotential deformation of SQCD, then our construction of $\widetilde{\omega}$ immediately generalizes upon substituting everywhere $\partial W/\partial M^{ij}$ for m_{ij}. The only important property of m which we used was the fact that it is annihilated by $\overline{\partial}$, which is always true for dW.

Renormalization group flow. To conclude, we consider how $\widetilde{\mathcal{O}}_\omega$ in (4.37) behaves under renormalization group flow. If we expand $\widetilde{\mathcal{O}}_\omega$ as a polynomial in m, then the term of degree k in m is given by

$$\begin{aligned}\mathcal{O}^{(n-k)}_\omega &= (-1)^k\binom{n-1}{k}\Lambda^{6-n}\,\left(\overline{M}M\right)^{-(n-k)}\,\epsilon^{i_1j_1\cdots i_nj_n}\,\times \\ &\qquad \times\overline{M}_{i_1j_1}\,m_{i_2j_2}\cdots m_{i_{k+1}j_{k+1}}\,\mathcal{O}_{i_{k+2}j_{k+2}}\cdots\mathcal{O}_{i_nj_n}\,, \\ \mathcal{O}_{ij} &\equiv M^{kl}\,\overline{D}\,\overline{M}_{ik}\cdot\overline{D}\,\overline{M}_{lj}\,.\end{aligned} \tag{4.43}$$

This operator $\mathcal{O}^{(n-k)}_\omega$ has the same form as the operator in (3.13) which appears in the theory with $n-k$ massless flavors.

We consider the limit in which k flavors have masses $m \gg \Lambda$. To integrate out these flavors, we restrict to the sublocus of $\mathcal{M}$ describing supersymmetric vacua in the massive theory, so that $m_{ik} M^{kj} = 0$ for all i, j (as follows from the F-term equations), and we simply omit from the operator $\widetilde{\mathcal{O}}_\omega$ any terms which involve the heavy quarks. The operator to which $\widetilde{\mathcal{O}}_\omega$ flows in the infrared is thus $\mathcal{O}_\omega^{(n-k)}$ in (4.43).

In particular, we can consider flowing to the theory with only one flavor. The operator to which $\widetilde{\mathcal{O}}_\omega$ flows is then given by

$$\mathcal{O}_\omega^{(1)} = (-1)^{(n-1)}\, \Lambda^{6-n} \left(\overline{M} M\right)^{-1} \epsilon^{i_1 j_1 \cdots i_n j_n}\, \overline{M}_{i_1 j_1}\, m_{i_2 j_2} \cdots m_{i_n j_n}\,. \tag{4.44}$$

As we see, $\mathcal{O}_\omega^{(1)}$ involves no fermions at all and represents a function on $\mathcal{M}$. Of course, this function is not holomorphic on all of $\mathcal{M}$.

However, if we restrict $\mathcal{O}_\omega^{(1)}$ to the sublocus of $\mathcal{M}$ describing supersymmetric vacua, then $\mathcal{O}_\omega^{(1)}$ is holomorphic. Indeed, this locus can be described by a single massless meson M, so the matrix structure disappears and $\overline{M}$ cancels out. On this locus, $\mathcal{O}_\omega^{(1)}$ can be written in terms of M as

$$\mathcal{O}_\omega^{(1)} = (-1)^{(n-1)}\, \Lambda^{6-n}\, \epsilon^{i_1 j_1 \cdots i_{n-1} j_{n-1}}\, m_{i_1 j_1} \cdots m_{i_{n-1} j_{n-1}}\, \frac{2}{M}\,. \tag{4.45}$$

In this expression, the pfaffian of the rank $2(n-1)$ minor of m appears, and an extra factor of two arises from the contraction of indices of $\overline{M}_{i_1 j_1}$. So, once ultraviolet and infrared scales are matched, $\mathcal{O}_\omega^{(1)}$ reproduces the nonperturbative superpotential in (4.32).

Acknowledgments

We would like to thank G. Moore and N. Seiberg for helpful suggestions on this work. CB would also like to thank A. Dymarsky, S. Murthy, P. Ouyang, D. Shih, and P. Svrček for stimulating conversations. Finally, CB would like to thank the participants of the 2004 Simons Workshop, where a preliminary version of this work was presented, for numerous comments and suggestions.

The work of CB is supported in part under National Science Foundation Grants No. PHY-0243680 and PHY-0140311, and the work of EW is supported in part under National Science Foundation Grant No. PHY-0070928. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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JHEP01(2005)056